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$q \to \infty$ limit of the quasitriangular WZW model

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Abstract

We study the $q \to \infty$ limit of the q-deformation of the WZW model
on a compact simple and simply connected target Lie group. We show
that the commutation relations of the $q \to \infty$ current algebra are un-
derlied by certain affine Poisson structure on the group of holomorphic
maps from the disc into the complexification of the target group. The
Lie algebroid corresponding to this affine Poisson structure can be in-
tegrated to a global symplectic groupoid which turns out to be nothing
but the phase space of the $q \to \infty$ limit of the q-WZW model. We
also show that this symplectic grupoid admits a chiral decomposition
compatible with its (anomalous) Poisson-Lie symmetries. Finally, we
dualize the chiral theory in a remarkable way and we evaluate the ex-
change relations for the $q \to \infty$ chiral WZW fields in both the original
and the dual pictures.

1 Introduction

The goal of the present paper is to study the $q \to \infty$ limit of the quasitriangular WZW model [9], which is the q-deformation of the standard WZW model [21] (the standard WZW model corresponds to the limit $q \to 1$). It will turn out that in the $q \to \infty$ limit the quasitriangular WZW model simplifies considerably, while enjoying chiral decomposability and other agreeable features of its finite q analogues. In particular, we shall find that the elliptic r-matrices, which characterize the chiral exchange relations, become trigonometric in the $q \to \infty$ limit. On the other hand, we shall show, rather remarkably, that inspite of this simplification, the symmetry pattern of the $q \to \infty$ model is richer and more intricate than in the case of finite q.

An important tool for analyzing the rich structure of the quasitriangular WZW model is the theory of affine Poisson groups and the associated concept of anomalous Poisson-Lie symmetry of dynamical systems. Affine Poisson groups have been introduced by Dazord and Sondaz [3] as generalizations of Poisson-Lie groups. To every affine Poisson bivector Π^* on a Lie group G^* there are associated two Poisson-Lie bivectors Π_L^*, Π_R^* on G^*. We shall denote by G_L and G_R the dual Poisson-Lie groups of the Poisson-Lie groups (G^*, Π_L^*) and (G^*, Π_R^*), respectively. In [13, 14], Lu has pointed out the following fact: if μ is a smooth Poisson map from a symplectic manifold P into (G^*, Π^*) then there exists a pair of (infinitesimal) Poisson-Lie symmetries of P with the symmetry groups G_L and G_R. If Π^* is itself a Poisson-Lie bivector, then $\Pi^* = \Pi_L^* = \Pi_R^*$ and $G_L = G_R \equiv G$. The Poisson map $\mu : P \to (G^*, \Pi^*)$ is then said to be an *equivariant* moment map of the pair of the G Poisson-Lie symmetries of P. Note that we still speak about the *pair* of symmetries because even in the equivariant case the action of the symmetry group $G_L = G$ on P need not coincide with the action of $G_R = G$ on P. If Π^* is Poisson but not a Poisson-Lie structure, then we say that the Poisson map $\mu : P \to (G^*, \Pi^*)$ is an *anomalous* moment map generating the pair of anomalous G_L and G_R Poisson-Lie symmetries of P.

We remark that even in the anomalous case, G_L and G_R may be isomorphic as the Lie groups. This happens for the quasitriangular WZW model if q is finite. (In the limit $q \to 1$, the symmetry structure of the model simplifies even more: not only G_L is isomorphic to G_R but also the $G_L = G$ action on

the phase space P of the model coincides with the $G_R = G$ action.) We shall show in this article that, at the opposite end of the range of the deformation parameter q, the full structural richness of the concept of the anomalous Poisson-Lie symmetry is realized. We mean by this that in the limit $q \to \infty$ the group G_L is no longer isomorphic to the group G_R. This fact leads to an interesting duality in the $q \to \infty$ WZW theory which interchanges the roles of the symmetry groups G_L and G_R in the description of the dynamics of the model.

The plan of the paper is as follows. In Sections 2 and 3, we review the elements of the theory of the affine Poisson groups and of the anomalous Poisson-Lie symmetries. In particular, we shall identify the moment maps of the anomalous Poisson-Lie symmetries of the symplectic grupoids integrating the affine Poisson groups. In Section 4, we turn our attention to the Poisson-Lie anomalies of loop group symmetries. The phase space of the $q \to \infty$ limit of the quasitriangular WZW model [9] will be shown to have the structure of the symplectic groupoid integrating certain affine Poisson structure on the group G^* whose elements are holomorphic maps from the disc into the complexification of the target group of the WZW model. We shall also establish the chiral decomposition of the $q \to \infty$ WZW theory and show that the chiral model is also Poisson-Lie symmetric. Moreover, we shall work out in detail the exchange (braiding) relations and $q \to \infty$ current algebra relations in the chiral sector. On the top of it, we shall describe also the remarkable duality of the chiral $q \to \infty$ WZW model which permits to express the symplectic structure of the model in terms of the two dual groups G_L and G_R naturally associated to the affine Poisson structure on G^*.

2 Affine Poisson groups

The affine Poisson groups were introduced by Dazord and Sondaz [3] and the basic facts about them can be found in [3, 13, 22]. We give here a short summary of some of the results contained in those papers. Note that all manifolds, maps or sections of bundles will be understood to be smooth, moreover, slightly abusing terminology, we shall often speak about e.g. vectors and forms instead of vector fields and form fields, respectively.

A manifold P equipped with a bivector Π is called Poisson if the following

bracket defines a Lie algebra commutator on the space of functions on P

$$\{x, y\} = \Pi(dx, dy).$$

Here x, y are any functions on P. Note that a map μ from another Poisson manifold P' into P is called Poisson if it intertwines the corresponding Poisson brackets, i.e.

$$\mu^*\{x, y\} = \{\mu^* x, \mu^* y\}'.$$

It is well-known [4, 16, 12, 3] that the Poisson bivector Π induces a Lie algebra structure also on the space consisting of 1-form fields on P. It is defined by the following formula [3]:

$$\{\alpha_1, \alpha_2\} = \mathcal{L}_{\Pi(\alpha_1, .)}\alpha_2 - \mathcal{L}_{\Pi(\alpha_2, .)}\alpha_1 - d(\Pi(\alpha_1, \alpha_2)), \quad (2.1)$$

where $\mathcal{L}_v \alpha$ denotes the Lie derivative of the 1-form α with respect to the vector field v.

Definition 1 Let $R(G^*)$ be the space of right-invariant 1-forms on a group manifold G^*. An affine Poisson structure on G^* is a Poisson bivector Π^* such that the bracket (2.1) defines the Lie algebra structure on $R(G^*)$. When the affine Poisson bivector vanishes at the unit element $e^* \in G^*$, the group (G^*, Π^*) is called the Poisson-Lie group.

$$\#$$

Lemma 1 A Poisson bivector Π^* on G^* defines an affine Poisson structure if and only if the bracket (2.1) defines a Lie algebra structure on the space $L(G^*)$ of left-invariant 1-forms on G^*.

$$\#$$

We remark that both spaces $R(G^*)$ and $L(G^*)$ can be naturally identified with the dual vector space $\mathcal{G}$ of the Lie algebra $\mathcal{G}^*$ of G^*. Thus the affine Poisson structure on G^* defines two (not necessarily) isomorphic Lie algebra structures on $\mathcal{G}$, we denote them $\mathcal{G}_L = (\mathcal{G}, [., .]_L)$ and $\mathcal{G}_R = (\mathcal{G}, [., .]_R)$.

Lemma 2 Let (G^*, Π^*) be an affine Poisson group and denote by M the value of the bivector Π^* at the group unit e^*. Then bivectors

$$\Pi_L^* \equiv \Pi^* - L_* M, \quad \Pi_R^* \equiv \Pi^* - R_* M$$

are both Poisson-Lie bivectors on G^* and a bivector

$$\Pi_{op}^* \equiv \Pi^* - R_* M - L_* M$$

3

is an affine Poisson bivector on G^*.

$$\#$$

The symbols L_* and R_* stand for the right and left translations from e^* onto the whole group manifold G^*. The bivectors Π_L^* and Π_R^* are referred to as the left and right Poisson-Lie structures associated to the affine Poisson structure Π^*; Π_{op}^* is called the opposed affine Poisson structure with respect to Π^*. It turns out that $(\Pi_{op}^*)_{op} = \Pi^*$ and Π_L^* (Π_R^*) is the right(left) associated Poisson-Lie structures to the affine Poisson structure Π_{op}^*.

Lemma 3 To every affine Poisson group (G^*, Π^*) it can be associated a Lie algebra $\mathcal{D}$ such that:
1) There exist three injective Lie algebra homomorphisms $\varsigma : \mathcal{G}^* \to \mathcal{D}$, $\varsigma_L : \mathcal{G}_L \to \mathcal{D}$ and $\varsigma_R : \mathcal{G}_R \to D$ such that $\mathcal{D} = \varsigma(\mathcal{G}^*) \dotplus \varsigma_L(\mathcal{G}_L)$ and also $\mathcal{D} = \varsigma(\mathcal{G}^*) \dotplus \varsigma_R(\mathcal{G}_R)$.
2) There is an Ad-invariant non-degenerate symmetric bilinear form $(.,.)_\mathcal{D}$ on D which vanishes when restricted to each of the subalgebras $\varsigma(\mathcal{G}^*), \varsigma_L(\mathcal{G}_L)$ and $\varsigma_R(\mathcal{G}_R)$.

$$\#$$

Note that the symbol $\dotplus$ means the direct sum of vector spaces but not necessarily the direct sum of Lie algebras (the latter denoted usually by $\oplus$).

The affine Poisson structure on a simply connected group G^* can be completely reconstructed from the Lie algebra $\mathcal{D}$ by considering a simply connected group D whose Lie algebra is $\mathcal{D}$. The explicite formula for Π^* is as follows

$$R_{g^{*-1}}\Pi^*(g^*)(\xi_1, \xi_2) = -(Ad_{\varsigma(g^*)^{-1}}\varsigma_L(\xi_1), p_R Ad_{\varsigma(g^*)^{-1}}\varsigma_L(\xi_2))_\mathcal{D}, \ g^* \in G^*, \xi_1, \xi_2 \in \mathcal{G}, \tag{2.2}$$

or, equivalently

$$L_{g^{*-1}}\Pi^*(g^*)(\xi_1, \xi_2) = -(Ad_{\varsigma(g^*)}\varsigma_R(\xi_2), p_L Ad_{\varsigma(g^*)}\varsigma_R(\xi_1))_\mathcal{D}, \quad g^* \in G^*, \quad \xi_1, \xi_2 \in \mathcal{G}. \tag{2.3}$$

Here $\varsigma : G^* \to D$ is the Lie group homomorphism integrating the inclusion map $\mathcal{G}^* \hookrightarrow \mathcal{D}$ and $p_R, p_L : \mathcal{D} \to \mathcal{D}$ are projectors with the kernel $\varsigma(\mathcal{G}^*)$ and the respective images $\varsigma_R(\mathcal{G}_R)$ and $\varsigma_L(\mathcal{G}_L)$. The group D is called the double of the affine Poisson group (G^*, Π^*).

Note that the projectors p_L, p_R and their respective adjoints p_L^*, p_R^* with respect to the bilinear form $(.,.)_\mathcal{D}$ can be all viewed as elements of $\mathcal{D} \otimes \mathcal{D}^*$.

Since the dual $\mathcal{D}^*$ can be identified with $\mathcal{D}$ via $(.,.)_{\mathcal{D}}$, we can view them also as elements of $\mathcal{D} \otimes \mathcal{D}$. In the latter case we denote them as P_L, P_R, P_L^* and P_R^*, respectively. Obviously, $(P_L - P_L^*)$ and $(P_R - P_R^*)$ are in $\mathcal{D} \wedge \mathcal{D}$.

Lemma 4 The following bivector on the group manifold D is Poisson:

$$\Pi_D = \frac{1}{2} L_*(P_L - P_L^*) + \frac{1}{2} R_*(P_R - P_R^*). \tag{2.4}$$

Moreover, the bivector Π_D is invertible on an open subset S of elements of D which can be simultaneously decomposed as products $\varsigma(u)\varsigma_L(v_L)$ and $\varsigma_R(v_R)\varsigma(\tilde{u})$ for some $u, \tilde{u} \in G^*$, $v_L \in G_L$ and $v_R \in G_R$.

$$\#$$

Remark: The symplectic manifold (S, Π_D) is nothing but the so-called global symplectic grupoid integrating the so-called Lie algebroid that corresponds to the affine Poisson structure (P, Π^*) (see [13, 15] for more details). We do not describe here the grupoid structure of S since we shall not need it in our study of the $q \to \infty$ limit of the quasitriangular WZW model. However, we shall continue to use the term symplectic grupoid in order not to give to the well-known structure a new name.

3 Poisson-Lie symmetry

In this paper, we shall use somewhat abbreviated terminology, by calling the Poisson-Lie symmetry of a Poisson manifold P what is usually referred to in the literature as the infinitesimal Poisson-Lie symmetry with moment map (cf. [13, 11, 5]). Moreover, we use the results of [13, 14] to rewrite the definition of this concept in the following form:

Definition 2 A Poisson manifold (P, Π) is Poisson-Lie symmetric with respect to an affine Poisson group (G^*, Π^*) if it exists a Poisson map $\mu :$ $(P, \Pi) \to (G^*, \Pi^*)$.

$$\#$$

If Π^* in e^* vanishes (does not vanish), the Poisson-Lie symmetry is called equivariant (anomalous). The Poisson map μ is referred to as the moment map and, as it is established in the following lemma, it permits *to express infinitesimal symmetry transformations in terms of Poisson brackets on the manifold P.*

Lemma 5 Let λ (ρ) be the left (right) invariant Maurer-Cartan form on the group manifold G^*. If $\mu : (P, \Pi) \to (G^*, \Pi^*)$ is a Poisson map, then the section $\Pi(., \mu^*\lambda) \in \mathcal{G}^* \otimes TP$ realizes a left $\mathcal{G}_L$ action on P and the section $\Pi(\mu^*\rho, .) \in \mathcal{G}^* \otimes TP$ a right $\mathcal{G}_R$ action on P.

$$\#$$

We observe that the Poisson-Lie symmetric manifolds always admit the *simultaneous* actions of *two symmetry Lie algebras*. However, in the equivariant case, the Lie algebras $\mathcal{G}_L$ and $\mathcal{G}_R$ are necessarily isomorphic and, moreover, if the affine Poisson group G^* is Abelian, even the $\mathcal{G}_L$ and $\mathcal{G}_R$ actions on P coincide.

Definition 3 We say that the double D of an affine Poisson group (G^*, Π^*) is proper if the images of the group homomorphisms $\varsigma, \varsigma_L, \varsigma_R$ are all simply connected and if the unit e_D of D is the unique element of the intersection $\varsigma_L(G_L) \cap \varsigma(G^*)$ and also of the intersection $\varsigma_R(G_R) \cap \varsigma(G^*)$.

$$\#$$

The properness of a double D means that every element $K \in S \subset D$ can be unambiguously decomposed in two ways: as $K = \varsigma_R(v_R)\varsigma(u)$, $v_R \in G_R$, $u \in G^*$ and as $K = \varsigma(\tilde{u})\varsigma_L(v_L)$, $v_L \in G_L$, $\tilde{u} \in G^*$. These decompositions obviously define four maps $\Lambda_L : S \to G^*$, $\Lambda_R : S \to G^*$, $\Xi_R : S \to G_L$, $\Xi_L : S \to G_R$ as follows:

$$\Lambda_L(K) \equiv \tilde{u}, \quad \Lambda_R(K) \equiv u^{-1}, \quad \Xi_L(K) = v_R, \quad \Xi_R(K) = v_L^{-1}. \qquad (3.1)$$

Theorem 1 The maps $\Lambda_L : (S, \Pi_D) \to (G^*, \Pi_{op}^*)$ and $\Lambda_R : (S, \Pi_D) \to (G^*, \Pi^*)$ are both Poisson.

Proof: We have to show that

$$\Lambda_{L*}\Pi_D(K) = \Pi_{op}^*(\Lambda_L(K)).$$

First we note that $p_L + p_L^* = p_R + p_R^* \in \mathcal{D} \otimes \mathcal{D}^*$ is nothing but the identity map from $\mathcal{D}$ to $\mathcal{D}$. It then follows that $P_L + P_L^* = P_R + P_R^* \in \mathcal{D} \otimes \mathcal{D}$ is Ad-invariant since it is the dual of the bilinear form $(.,.)_\mathcal{D}$. Thus from the equality $L_*(P_L + P_L^*) = R_*(P_R + P_R^*)$ we deduce

$$\Pi_D = L_*P_L - R_*P_R^*.$$

From the very definition of the map Λ_L it follows that

$$\Lambda_{L*}R_{\varsigma_L(v_L)*}w = \Lambda_{L*}w, \quad \Lambda_{L*}L_{\varsigma(u)*}w = L_{u*}\Lambda_{L*}w \qquad (3.2)$$

6

for a whatever vector $w \in T_K S$ and whatever elements $v_L \in G_L$ and $u \in G^*$. (Here e.g. L_{K*} means the left transport by the element $K \in D$.) By using the relations (3.2), we easily arrive at

$$\Lambda_{L*} L_{K*} P_L = L_{\Lambda_L(K)*} \Lambda_{L*} Ad_{\varsigma_L(\Xi_R^{-1}(K))} P_L, \ \Lambda_{L*} R_{K*} P_R^* = L_{\Lambda_L(K)*} \Lambda_{L*} Ad_{\varsigma(\Lambda_L^{-1}(K))} P_R^*.$$
$$(3.3)$$

With the help of Eqs. (3.3), we infer for any $\xi_1, \xi_2 \in \mathcal{G}$

$$< \Lambda_{L*} \Pi_D(K), L_{\Lambda_L(K)^{-1}}^*(\xi_1 \otimes \xi_2) > = < \Lambda_{L*} \Big(Ad_{\varsigma_L(\Xi_R^{-1}(K))} P_L - Ad_{\varsigma(\Lambda_L^{-1}(K))} P_R^* \Big), \xi_1 \otimes \xi_2 > =$$

$$= 0 - (Ad_{\varsigma(\Lambda_L(K))} \varsigma_L(\xi_2), p_R Ad_{\varsigma(\Lambda_L(K))} \varsigma_L(\xi_1))_\mathcal{D} = < \Pi_{op}^*(\Lambda_L(K)), L_{\Lambda_L(K)^{-1}}^*(\xi_1 \otimes \xi_2) > .$$

In a similar manner, we show that

$$\Lambda_{R*} \Pi_D(K) = \Pi^*(\Lambda_R(K)).$$

$$\#$$

Remark: We note that our study of the structure of the symplectic grupoids of affine Poisson groups is similar in spirit to the study of the symplectic groupoids of Poisson-Lie groups in [1].

Because the moment map Λ_R is Poisson with respect to the affine Poisson structure Π^*, it simultaneously realizes the right $\mathcal{G}_R$ Poisson-Lie symmetry and the left $\mathcal{G}_L$ Poisson-Lie symmetries. However, because the moment map Λ_L is Poisson with respect to the *opposed* affine Poisson structure Π_{op}^*, it simultaneously realizes the *left* $\mathcal{G}_R$ Poisson-Lie symmetry and the *right* $\mathcal{G}_L$ Poisson-Lie symmetry. In the applications of our general theory, presented in the next chapter, we shall need the explicit formulae for the right Poisson-Lie symmetries of S induced by the moment maps Λ_L and Λ_R.

Theorem 2 The section $\Pi(\Lambda_L^* \rho, .)$ generates the infinitesimal version of the natural right G_R action on D: $(g_R, K) \to \varsigma_R(g_R^{-1})K$ for $g_R \in G_R$ and $K \in D$. Similarly, the section $\Pi(\Lambda_R^* \rho, .)$ generates the infinitesimal version of the natural right G_L action on D: $(g_R, K) \to K \varsigma_L(g_L)$ for $g_L \in G_L$ and $K \in D$.

Proof: Consider a point $K \in S$ and elements $\tilde{\eta} \in \mathcal{D}^*$, $\xi \in \mathcal{G}$. We set $\rho_\xi \equiv < \rho, \xi > = R^* \xi$ and write

$$< \Pi_D, \Lambda_L^* \rho_\xi \otimes R_{K^{-1}}^* \tilde{\eta} > = < L_{K*} P_L - R_{K*} P_R^*, \Lambda_L^* \rho_\xi \otimes R_{K^{-1}}^* \tilde{\eta} > =$$

7

$$=< P_L, (R_{\Lambda_L(K)^{-1}}\Lambda_L L_K)^* \xi \otimes (L_K R_{K^{-1}})^* \tilde{\eta} > - < P_R^*, (R_{\Lambda_L(K)^{-1}}\Lambda_L R_K)^* \xi \otimes \tilde{\eta} > .$$

Denote by η the element of $\mathcal{D}$ which corresponds to $\tilde{\eta} \in \mathcal{D}^*$ upon the identification by the bilinear form $(.,.)_{\mathcal{D}}$. Then

$$< \Pi_D, \Lambda_L^* \rho_\xi \otimes R_{K^{-1}}^* \tilde{\eta} >=< (R_{\Lambda_L(K)^{-1}}\Lambda_L L_K)^* \xi, p_L Ad_K \eta > - < (R_{\Lambda_L(K)^{-1}}\Lambda_L L_K)^* \xi, p_R^* \eta > .$$

Taking into account Eqs. (3.3), we infer

$$< \Pi_D, \Lambda_L^* \rho_\xi \otimes R_{K^{-1}}^* \tilde{\eta} >=$$

$$= (\varsigma_R(\xi), Ad_{\varsigma(\Lambda_L(K))} p_L^* \left(Ad_{\varsigma_L(\Xi_R(K)^{-1})} p_L Ad_K \eta - Ad_{\varsigma(\Lambda_L(K)^{-1})} p_R^* \eta \right))_{\mathcal{D}} =$$

$$= -(\varsigma_R(\xi), p_R^* \eta)_{\mathcal{D}} = - < \tilde{\eta}, \varsigma_R(\xi) >= - < R_{K*}\varsigma_R(\xi), R_{K^{-1}}^* \tilde{\eta} > .$$

We thus arrive to the announced conclusion

$$\Pi_D(\Lambda_L^* \rho_\xi, .) = -R_* \varsigma_R(\xi). \tag{3.4}$$

In a similar manner, we show that

$$\Pi_D(\Lambda_R^* \rho_\xi, .) = L_* \varsigma_L(\xi). \tag{3.5}$$

$$\#$$

In Section 4, we shall need an explicit formula for the symplectic form ω_D corresponding to the Poisson bivector Π_D on a *proper* double D.

Theorem 3 Let (D, Π_D) be a proper double and $S \subset D$ be the corresponding symplectic grupoid. Denote by ρ, ρ_L and ρ_R the right-invariant Maurer-Cartan forms on the respective groups G^*, G_L and G_R. The symplectic form ω_S on S is then given by the following formula

$$\omega_S = \frac{1}{2}(\Lambda_L^* \rho \overset{\wedge}{,} \Xi_L^* \rho_R)_{\mathcal{D}} + \frac{1}{2}(\Lambda_R^* \rho \overset{\wedge}{,} \Xi_R^* \rho_L)_{\mathcal{D}}. \tag{3.6}$$

Proof: Choose a basis t_i of $\mathcal{G}^*$ and the basis T_L^i of $\mathcal{G}_L$ and T_R^i of $\mathcal{G}_R$ such that

$$(t_i, T_L^j) = \delta_i^j, \quad (t_i, T_R^j) = \delta_i^j. \tag{3.7}$$

The form ω_S can be then rewritten as

$$\omega_S = \frac{1}{2}(\Lambda_L^* \rho, T_R^i)_{\mathcal{D}} \wedge (\Xi_L^* \rho_R, t_i)_{\mathcal{D}} + \frac{1}{2}(\Lambda_R^* \rho, T_L^i)_{\mathcal{D}} \wedge (\Xi_R^* \rho_L, t_i)_{\mathcal{D}}.$$

8

We are going to show that the 2-form ω_S is the inverse of the Poisson bivector Π_D restricted to S.

Consider a point $K \in S$ and four linear subspaces of the tangent space $T_K S$ defined as $S_L = L_{K*}\mathcal{G}_L$, $S_R = R_{K*}\mathcal{G}_R$, $\tilde{S}_L = L_{K*}\mathcal{G}^*$ and $\tilde{S}_R = R_{K*}\mathcal{G}^*$. At every $K \in S$ (but not necessarily at every $K \in D$!) the tangent space $T_K S$ can be decomposed as $T_K S = S_L + \tilde{S}_R$ and $T_K P = \tilde{S}_L + S_R$, respectively. We introduce a projector $\Pi_{L\tilde{R}}$ on $\tilde{S}_R$ with a kernel S_L, a projector $\Pi_{\tilde{L}R}$ on S_R with a kernel $\tilde{S}_L$, a projector $\Pi_{R\tilde{R}}$ on $\tilde{S}_R$ with a kernel S_R and a projector $\Pi_{\tilde{L}L}$ on S_L with a kernel $\tilde{S}_L$. Note that the first subscript stands for the kernel and the second for the image. Then we have

$$< (\Lambda_L^* \rho, T_R^i)_{\mathcal{D}}, t >= (R_{K*}T_R^i, \Pi_{L\tilde{R}}t)_{\mathcal{D}}, \qquad (3.8)$$

$$< (\Xi_L^* \rho_R, t_i)_{\mathcal{D}}, t >= (R_{K*}t_i, \Pi_{\tilde{L}R}t)_{\mathcal{D}}, \qquad (3.9)$$

$$< (\Lambda_R^* \rho, T_L^i)_{\mathcal{D}}, t >= -(L_{K*}T_L^i, \Pi_{R\tilde{L}}t)_{\mathcal{D}}, \qquad (3.10)$$

$$< (\Xi_R^* \rho_L, t_i)_{\mathcal{D}}, t >= -(L_{K*}t_i, \Pi_{\tilde{R}L}t)_{\mathcal{D}}, \qquad (3.11)$$

where t is a vector at a point $K \in S$.

Let us show how to demonstrate (3.8-11) on an example (3.8). For $K \in S$, the vectors $L_{K*}T_L^i$, $R_{K*}t_i$ form the basis of the tangent space $T_K S$. Thus it is sufficient to prove (3.8) for t being one of the elements of the basis of $T_K S$. For $t = L_{K*}T_L^j$, it is obvious that the r.h.s. of (3.8) vanishes. On the other hand, knowing that $\Lambda_L(K e^{sT_L^j}) = \Lambda_L(K)$, we can evaluate the l.h.s.:

$$< (\Lambda_L^* \rho, T_R^i)_{\mathcal{D}}, L_{K*}T_L^j >=< (\rho, T_R^i)_{\mathcal{D}}, \Lambda_{L*}(L_{K*}T_L^j) >= 0.$$

For $t = R_{K*}t_j$, the r.h.s. of (3.8) gives

$$(R_{K*}T_R^i, \Pi_{L\tilde{R}}R_{K*}t_j)_{\mathcal{D}} = (R_{K*}T_R^i, R_{K*}t_j)_{\mathcal{D}} = \delta_j^i.$$

On the other hand, knowing that $\Lambda_L(e^{st_j}K) = e^{st_j}\Lambda_L(K)$, we can evaluate the l.h.s.:

$$< (\Lambda_L^* \rho, T_R^i)_{\mathcal{D}}, R_{K*}t_j >=< (\rho, T_R^i)_{\mathcal{D}}, \Lambda_{L*}R_{K*}t_j >=$$

$$=< (\rho, T_R^i)_{\mathcal{D}}, R_{\Lambda_L(K)*}t_j >= (R_{\Lambda_L^{-1}(K)*}R_{\Lambda_L(K)*}t_j, T_R^i)_{\mathcal{D}} = (t_j, T_R^i)_{\mathcal{D}} = \delta_j^i.$$

9

By using the relations (3.8-11), we can evaluate the form ω_S on any two vectors $t, u \in T_K P$ in terms of the projectors:

$$2\omega_S(t, u) =$$

$$= (R_{K*}T_R^i, \Pi_{L\tilde{R}}t)_{\mathcal{D}}(R_{K*}t_i, \Pi_{\tilde{L}R}u)_{\mathcal{D}} + (L_{K*}T_L^i, \Pi_{R\tilde{L}}t)_{\mathcal{D}}(L_{K*}t_i, \Pi_{\tilde{R}L}u)_{\mathcal{D}} - (t \leftrightarrow u) =$$

$$= (\Pi_{L\tilde{R}}t, \Pi_{\tilde{L}R}u)_{\mathcal{D}} + (\Pi_{R\tilde{L}}t, \Pi_{\tilde{R}L}u)_{\mathcal{D}} - (t \leftrightarrow u).$$

Here $(.,.)_{\mathcal{D}}$ is the bi-invariant metric at the point K. By realizing that it holds

$$(t, \Pi_{\tilde{L}R}u)_{\mathcal{D}} = (\Pi_{R\tilde{L}}t, \Pi_{\tilde{L}R}u)_{\mathcal{D}} = (\Pi_{R\tilde{L}}t, u)_{\mathcal{D}},$$

$$\Pi_{\tilde{L}R} + \Pi_{R\tilde{L}} = Id,$$

we finally arrive at

$$\omega_S(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}}.$$

Now we can easily show that ω_S is the symplectic form corresponding to the Poisson structure Π_D restricted to S. First of all, we remark that Π_D can be written also as

$$\Pi_D = L_*(T_L^i \otimes t_i) - R_*(t_i \otimes T_R^i).$$

Then we conclude

$$\Pi_D(., \omega_S(., u)) =$$

$$= L_{K*}T_L^i(L_{K*}t_i, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}} - R_{K*}t_i(R_{K*}T_R^i, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}} =$$

$$= (\Pi_{\tilde{L}L} - \Pi_{R\tilde{R}})(\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u = (\Pi_{R\tilde{R}}\Pi_{L\tilde{R}} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}} + \Pi_{\tilde{L}L}\Pi_{\tilde{L}R})u =$$

$$= (\Pi_{L\tilde{R}} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}} + \Pi_{\tilde{L}L})u = (\Pi_{L\tilde{R}} + \Pi_{\tilde{R}L})u + (\Pi_{\tilde{L}L} - \Pi_{\tilde{R}L} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}})u =$$

$$= (\Pi_{L\tilde{R}} + \Pi_{\tilde{R}L})u = u. \qquad (3.12)$$

From the equation (3.12), we learn that the form ω_S is invertible and its inverse is nothing but the Poisson bivector Π_D restricted to S. From this it also follows, by the way, that ω_S is closed hence symplectic.

$$\#$$

4 $q \to \infty$ WZW model on a compact group

4.1 $q \to \infty$ limit of a twisted Heisenberg double

We remind that the quasitriangular WZW model (or the q-WZW model for short) is the q-deformation of the standard WZW model. It exhibits the Poisson-Lie symmetries with respect to two different (chiral) actions of the polynomial loop group $L_{pol}K$ on the phase space of the model and those symmetries become Hamiltonian in the limit $q \to 1$. We believe that this paper would become too long if we attempted to review here the full structure of the q-WZW model, and, in particular, the details of the limit $q \to 1$. Thus, for the sake of economy, we shall point out only few facts concerning the finite q that are indispensable for the good understanding of the $q \to \infty$ limit. The reader, who will feel a need to learn more about the situation for the finite q, can consult [9, 10].

The symplectic structure of the quasitriangular WZW model for finite q is that of the twisted Heisenberg double of the complexified polynomial loop group $L_{pol}K^{\mathbf{C}}$ [10]. The concept of the twisted Heisenberg double (D, κ) is due to Semenov-Tian-Shansky [19] and it can be defined for every automorphism κ of the double D of a Poisson-Lie group G^* such that κ preserves the canonical invariant bilinear form $(., .)_{\mathcal{D}}$ on $Lie(D)$. Thus (D, κ) is a Poisson manifold with the Poisson bivector Π_D^κ defined as follows

$$\Pi_D^\kappa = \frac{1}{2}L_*(P - P^*) + \frac{1}{2}R_*\kappa_*(P - P^*). \qquad (4.1)$$

In this formula, the elements $P, P^* \in \mathcal{D} \wedge \mathcal{D}$ are defined as in Lemma 4, but we abandon the subscripts L, R in P_L, P_R since for the Poisson-Lie group $P_L = P_R \equiv P$.

Remark. The notions of the symplectic grupoid and of the twisted Heisenberg double are not quite equivalent, inspite of certain similarity between their respective Poisson structures (2.4) and (4.1). However, in some circumstances they become equivalent; e.g. if the automorphism κ preserves the subgroup $\varsigma(G^*)$. In this special case, the twisted Heisenberg double (D, κ) of a Poisson-Lie group G^* can be interpreted as the symplectic grupoid of certain κ-dependent affine Poisson structure on G^* for which the original Poisson-Lie structure on G^* is the left-associated Poisson-Lie structure. This

particular situation takes place for the finite q quasitriangular WZW model, however, as we shall see, in the limit $q \to \infty$ this is no more the case and the usage of the symplectic grupoids instead of the twisted Heisenberg doubles becomes essential.

Let us view a connected simply connected simple compact Lie group K as a subgroup of the group of unitary matrices of order n. Following [18], the group of polynomial loops $L_{pol}K$ consists of matrix valued functions $\gamma(\sigma) \in K$, defined on the circle parametrized by an angular variable σ, for which there exist non-negative integers p_+, p_- and $n \times n$-matrices γ_k such that

$$\gamma(\sigma) = \sum_{k=-p_-}^{k=p_+} \gamma_k e^{ik\sigma}. \tag{4.2}$$

Denote by $(.,.)_{\mathcal{K}}$ the negative-definite Ad-invariant Killing form on the Lie algebra $\mathcal{K} \equiv Lie(K)$ and define a non-degenerate Ad-invariant bilinear form $(.|.)$ on $Lie(L_{pol}K)$ by the following formula

$$(\alpha|\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma(\alpha(\sigma), \beta(\sigma))_{\mathcal{K}}. \tag{4.3}$$

Denote by D the complexification $L_{pol}K^{\mathbf{C}}$ of $L_{pol}K$ and view it as the *real* group. We note that the elements of D are also of the form (4.2), however, for each σ, the result of the summation is in $K^{\mathbf{C}}$ and not just in K. Let G^* be a subgroup of D, consisting of the elements of $L_{pol}K^{\mathbf{C}}$ of the form (4.2), for which $p_- = 0$ and $\gamma_0 \in AN$. Here AN is the subgroup of $K^{\mathbf{C}}$ defined by the Iwasawa decomposition $K^{\mathbf{C}} = KAN$. We remark that the elements of G^* can be viewed as the boundary values of the holomorphic maps $\tilde{\gamma} : \{z \in \mathbf{C} : |z| \leq 1\} \to K^{\mathbf{C}}$. The following factorization lemma was proved in [18] and it has a crucial importance for the present paper:
Lemma 6 Any element $l \in D$ can be factorized uniquely as $l = uv$, $u \in G^*$ and $v \in L_{pol}K$ and the product map $G^* \times L_{pol}K \to D$ is a diffeomorphism.
$$\#$$
Remark. In what follows, we shall suppress the symbols ς, ς_L and ς_R, standing for the group homomorphisms introduced in Sec 2. We do it in order to avoid cumbersome formulae and we hope that the reader will easily reconstitute them from the context.

Let us introduce a nondegenerate bilinear form $(.,.)_{\mathcal{D}}$ on $\mathcal{D} \equiv Lie(D)$ defined as

$$(x,y)_{\mathcal{D}} = \frac{1}{\epsilon} Im(x|y), \quad x,y \in \mathcal{D}. \qquad (4.4)$$

Here $(.|.)$ is just the bilinear form (4.3) naturally extended to $Lie(L_{pol}K^{\mathbf{C}})$, Im stands for the imaginary part and $\epsilon \equiv \ln q$ is the deformation parameter. It is not difficult to establish that the restrictions of the bilinear form $(.,.)_{\mathcal{D}}$ on $Lie(L_{pol}K)$ and on $Lie(G^*)$ both vanish. For $Lie(L_{pol}K)$, it follows from the fact that the bilinear form $(.|.)$ is real when restricted to $Lie(L_{pol}K)$ and for $Lie(G^*)$, it follows from the fact that the integral of the product of two functions containing only non-negative Fourier modes vanishes unless both functions contain a zero mode. However, the zero mode part of $Lie(G^*)$ is $Lie(AN)$ and the restriction of $(.,.)_{\mathcal{D}}$ on $Lie(AN)$ vanishes (cf. Sec 4.4.1 of [9]).

The data $D = L_{pol}K^{\mathbf{C}}$, G^*, $L_{pol}K$ and $(.,)_{\mathcal{D}}$, that we have just introduced, define a Poisson-Lie structure on G^*. This follows from Lemma 3 and Eqs. (2.2), (2.3), for the special case when $\mathcal{G}_L = \mathcal{G}_R = Lie(L_{pol}K)$. Let κ denote the following automorphism κ of D, preserving the bilinear form $(.,.)_{\mathcal{D}}$:

$$\kappa(l)(\sigma) = l(\sigma + ik\epsilon), \quad l \in L_{pol}K^{\mathbf{C}}. \qquad (4.5)$$

The double (D, κ) is then nothing but the twisted Heisenberg double of the Poisson-Lie group G^* and the Poisson bivector (4.1) determines the symplectic structure of the quasitriangular WZW model for finite q. Note also that the integer parameter k, appearing in (4.5), becomes the level of the standard WZW model in the limit $q \to 1$.

In what follows, it will be useful to work with an appropriate choice of the basis of $\mathcal{D} = Lie(L_{pol}K^{\mathbf{C}})$. We normalize the extension of the Killing-Cartan form $(.,.)_{\mathcal{K}}$ on $K^{\mathbf{C}}$ in such a way that the square of the length of the longest root is equal to two. We pick an orthonormal Hermitian basis H^μ in the Cartan subalgebra $\mathcal{H}^{\mathbf{C}}$ of $\mathcal{K}^{\mathbf{C}}$ with respect to the Killing Cartan form $(.,.)_{\mathcal{K}}$. Consider the root space decomposition of $\mathcal{K}^{\mathbf{C}}$:

$$\mathcal{K}^{\mathbf{C}} = \mathcal{H}^{\mathbf{C}} \bigoplus (\oplus_{\alpha \in \Phi} \mathbf{C}E^\alpha),$$

where α runs over the space Φ of all roots $\alpha \in \mathcal{H}^{*\mathbf{C}}$. The step generators E^α fulfil

$$[H^\mu, E^\alpha] = \alpha(H^\mu)E^\alpha, \quad (E^\alpha)^\dagger = E^{-\alpha};$$

$$[E^\alpha, E^{-\alpha}] = \alpha^\vee, \quad [\alpha^\vee, E^{\pm\alpha}] = \pm 2E^{\pm\alpha}, \quad (E^\alpha, E^{-\alpha})_\kappa = \frac{2}{|\alpha|^2}.$$

The element $\alpha^\vee \in \mathcal{H}^{\mathbf{C}}$ is called the coroot of the root α. Thus the (ordinary Cartan-Weyl) basis of the complex Lie algebra $\mathcal{K}^{\mathbf{C}}$ is (H^μ, E^α), $\alpha \in \Phi$. The affine Cartan-Weyl basis of $Lie(L_{pol}K^{\mathbf{C}})$ is now formed by the elements of the form

$$E^\alpha e^{in\sigma} \equiv E_n^\alpha, \quad n \in \mathbf{Z}, \qquad H^\mu e^{in\sigma} \equiv H_n^\mu, \quad n \in \mathbf{Z}.$$

We call the elements E_n^α, H_n^μ the affine step generators with the exception of H_0^μ which will be called the affine Cartan generators. The commutation relations in the affine Cartan-Weyl basis easily follow from those of the ordinary Cartan-Weyl basis of $\mathcal{K}^{\mathbf{C}}$. The automorphism κ of D defined by (4.5) descends to the following automorphism of $Lie(D)$:

$$\kappa_* E_n^\alpha = q^{-nk} E_n^\alpha, \quad n \in \mathbf{Z}, \qquad \kappa_* H_n^\mu = q^{-nk} H_n^\mu, \quad n \in \mathbf{Z}. \qquad (4.6)$$

In what follows, we shall often denote a generic affine step generator as $E^{\hat{\alpha}}$, where $\hat{\alpha} \in \hat{\Phi}$ stands for the corresponding labels (α, n) or $(\mu, n \neq 0)$. If $\hat{\alpha}$ is such that α, μ are arbitrary and $n > 0$, or $\alpha > 0$ and $n = 0$, we say that $\hat{\alpha} > 0$.

A basis of the Lie subalgebra $Lie(L_{pol}K) \subset Lie(L_{pol}K^{\mathbf{C}})$ can be then chosen as $(T_L^\mu, B_L^{\hat{\alpha}}, C_L^{\hat{\alpha}})$, $\hat{\alpha} > 0$ where

$$T_L^\mu = iH^\mu, \quad B_L^{\hat{\alpha}} = \frac{i}{\sqrt{2}}(E^{\hat{\alpha}} + E^{-\hat{\alpha}}), \quad C_L^{\hat{\alpha}} = \frac{1}{\sqrt{2}}(E^{\hat{\alpha}} - E^{-\hat{\alpha}}). \qquad (4.7)$$

Here by $-\hat{\alpha}$ we mean $(-\alpha, -n)$ for $\hat{\alpha} = (\alpha, n)$ and $(\mu, -n)$ for $\hat{\alpha} = (\mu, n)$. The meaning of the subscript L will become clear soon.

In a similar manner, a basis of $Lie(G^*)$ will be denoted as $(t_\mu, b_{\hat{\alpha}}, c_{\hat{\alpha}})$, $\hat{\alpha} > 0$ and it reads:

$$t_\mu = H^\mu, \quad b_{\hat{\alpha}} = \frac{|\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}, \quad c_{\hat{\alpha}} = -i\frac{|\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}.$$

Note that for the roots of the type $\hat{\alpha} = (\mu, n)$, we set $|\hat{\alpha}|^2 = 2$.

With the help of the respective basis of $Lie(L_{pol}K)$ and of $Lie(G^*)$, the Poisson structure (4.1) of the quasitriangular WZW model for a finite q can be rewritten as

$$\Pi_D^\kappa = L_*\left(T_L^\mu \otimes t_\mu + B_L^{\hat{\alpha}} \otimes b_{\hat{\alpha}} + C_L^{\hat{\alpha}} \otimes c_\alpha\right) - R_*\kappa_*\left(t_\mu \otimes T_L^\mu + b_{\hat{\alpha}} \otimes B_L^{\hat{\alpha}} + c_{\hat{\alpha}} \otimes C_L^{\hat{\alpha}}\right).$$

In this expression, the limit $q \to \infty$ can be directly performed with the help of Eqs. (4.6) and (4.7). The result is as follows

$$\Pi_D^\infty = L_*\left(T_L^\mu \otimes t_\mu + B_L^{\hat{\alpha}} \otimes b_{\hat{\alpha}} + C_L^{\hat{\alpha}} \otimes c_\alpha\right) - R_*\left(t_\mu \otimes T_R^\mu + b_{\hat{\alpha}} \otimes B_R^{\hat{\alpha}} + c_{\hat{\alpha}} \otimes C_R^{\hat{\alpha}}\right), \quad (4.8)$$

where for $n > 0$ we define

$$B_R^{\hat{\alpha}} = \frac{i}{\sqrt{2}}E^{-\hat{\alpha}}, \quad C_R^{\hat{\alpha}} = -\frac{1}{\sqrt{2}}E^{-\hat{\alpha}}, \ \hat{\alpha} > 0$$

and for $n = 0$

$$T_R^\mu = iH^\mu, \quad B_R^{(\alpha,0)} = \frac{i}{\sqrt{2}}(E_0^\alpha + E_0^{-\alpha}), \quad C_R^{(\alpha,0)} = \frac{1}{\sqrt{2}}(E_0^\alpha - E_0^{-\alpha}), \ \alpha > 0.$$

With the help of the theory of the affine Poisson groups, it is easy to understand the structure of the Poisson bivector Π_D^∞. In fact, it turns out that it defines a symplectic structure of the symplectic grupoid S_∞ corresponding to certain affine Poisson structure on the group G^*. To see that, first we introduce a subgroup G_R of D, consisting of the elements of $L_{pol}K^{\mathbf{C}}$ of the form (4.2), for which $p_+ = 0$ and $\gamma_0 \in K$. We remark that the elements of G_R can be viewed as the boundary values of the holomorphic maps $\tilde{\gamma} : \{z \in \mathbf{C} \cup \infty : |z| \geq 1\} \to K^{\mathbf{C}}$. It is not difficult to see that the elements $(T_R^\mu, B_R^{\hat{\alpha}}, C_R^{\hat{\alpha}})$, $\hat{\alpha} > 0$ form the basis of the Lie algebra $Lie(G_R)$. The data $D = L_{pol}K^{\mathbf{C}}$, G^*, $G_L \equiv L_{pol}K$ and G_R define the affine Poisson structure on G^*, given by Eqs. (2.2),(2.3). Finally, by working out the corresponding bivector (2.4) in the chosen basis on $\mathcal{G}_L \equiv Lie(L_{pol}K)$, $\mathcal{G}_R \equiv Lie(G_R)$ and $\mathcal{G}^* \equiv Lie(G^*)$, we obtain the formula (4.8).

4.2 $\ q \to \infty$ current algebras

So far we have worked out the $q \to \infty$ limit of the Poisson structure of the q-WZW model and we have established that it coincides with the Poisson

structure of the symplectic grupoid of certain affine Poisson structure on the group G^*. In order to establish the Poisson-Lie symmetries of this symplectic grupoid, we wish to show that the double D is proper in the sense of Definition 3. This is indeed the case because, first of all, the subgroups G_L, G_R and G^* are all simply connected (cf. [18]), secondy, because G_L intersects G^* only at the unit element $e_D \in D$ (cf. Lemma 6) and, finally, because any common element of G_R and G^* defines a global holomorphic map from the Riemann sphere into $K^{\mathbf{C}}$ and, therefore, it must be σ-independent element of D belonging at the same time to K and to AN. Obviously, only such element is again e_D. We thus see that the hypothesis of Theorem 1 are satisfied and the $q \to \infty$ limit of the q-WZW model enjoys a rich structure of the Poisson-Lie symmetries defined with the help of the moment maps Λ_L, Λ_R defined on the phase space $S_\infty \equiv G_R G^* \cap G_L G^* \subset D$. The most interesting feature of this symmetry structure is the fact that the right and the left symmetry Lie algebras $\mathcal{G}_R$ and $\mathcal{G}_L$ are not isomorphic. This is a genuinely new phenomenon arising in the $q \to \infty$ WZW model which has no analogue for any finite q and, as we shall see, it is in the origin of an interesting duality of the chiral version of the $q \to \infty$ model.

Before studying the issue of the $\mathcal{G}_L \leftrightarrow \mathcal{G}_R$ duality, we have to work out several general formulae in our loop group context, in particular we shall need the affine Poisson structures Π^* and Π^*_{op} on the group G^*. The bivector Π^*, with respect to which the map Λ_R is Poisson, is given by the general formula (2.2), while its opposed bivector Π^*_{op}, with respect to which the map Λ_L is Poisson, is given by the same formula with the subscripts L and R interchanged. As it is costumary in the studies of WZW models, we shall characterize the Poisson bivectors Π^* and Π^*_{op} by Poisson brackets of a set of particular coordinate functions on the group G^* called the "Kac-Moody functions". Actually, these Kac-Moody functions, pull-backed by the moment maps Λ_L and Λ_R to the phase space $S_\infty = G_R G^* \cap G^* G_L = G_R G^*$, are the observables of the WZW model that generate the symmetries via the Poisson brackets on S_∞, and they are commonly referred to as the "Kac-Moody" currents. Actually, for $q = 1$, the Poisson brackets of the Kac-Moody currents give the standard current algebra commutators; for a finite q, they give the q-current algebra relations described in [9] and for $q \to \infty$ they give ∞-current algebra relations that we are going to work out.

The Kac-Moody functions on the group G^*, the Poisson brackets of which

determine the bivectors Π^* and Π^*_{op}, are obviously the same as for the finite q. To describe them, we pick an irreducible unitary representation Υ of the compact group K and consider it as the representation of the complexified group $K^{\mathbf{C}}$. We pick also a point on the loop characterized by a particular value of the angle coordinate σ. The data (Υ, σ) define $r \times r$ functions on $LK^{\mathbf{C}}$, where r is the dimension of the representation Υ. In words, these functions are defined as follows: take an element $l \in L_{pol}K^{\mathbf{C}}$, consider the element $l(\sigma) \in K^{\mathbf{C}}$ and, finally, matrix elements ij of the element $l(\sigma)$ in the representation Υ.

The functions Υ^{ij}_σ are holomorphic. Since we regard G^* as the *real* group, the Poisson brackets of just holomorphic functions cannot fully describe the Poisson structures Π^* and Π^*_{op}. In fact, we must also consider the antiholomorhic functions $(\Upsilon^\dagger_\sigma)^{-1}$ and calculate their brackets with the holomorphic ones. The calculation is considerably simplified if one uses the notation of the matrix valued Poisson brackets [6, 9]. Thus, if V is a vector space and E and F two $End(V)$-valued functions on G^*, we introduce a matrix Poisson bracket $\{E \overset{\otimes}{,} F\}$ as the $(End(V) \otimes End(V))$-valued function on G^* defined as

$$\{E \overset{\otimes}{,} F\}^{ik,jl} = \{E^{ij}, F^{kl}\}.$$

We wish to calculate Poisson brackets $\{.,.\}^*$ or $\{.,.\}^*_{op}$ of the matrix valued functions Υ_σ, $(\Upsilon^\dagger_\sigma)^{-1}$ restricted on G^*. The calculation is not difficult, it is based on the following obvious relations

$$< L_* t, d\Upsilon^{ij}_\sigma >= (\Upsilon_\sigma)^{ik}\Upsilon(t)^{kj}; \tag{4.9}$$

$$< R_* t, d\Upsilon^{ij}_\sigma >= \Upsilon(t)^{ik}\Upsilon^{kj}_\sigma, \tag{4.10}$$

for every $t \in \mathcal{D}$. The result reads:

$$\{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}^* = r(\sigma - \sigma')\Big(\Upsilon_\sigma \otimes \Upsilon_{\sigma'}\Big) - \Big(\Upsilon_\sigma \otimes \Upsilon_{\sigma'}\Big)r(\sigma - \sigma'). \tag{4.11}$$

$$\{\Upsilon^{\dagger}_\sigma{}^{-1} \overset{\otimes}{,} \Upsilon^{\dagger}_{\sigma'}{}^{-1}\}^* = r(\sigma - \sigma')\Big(\Upsilon^{\dagger}_\sigma{}^{-1} \otimes \Upsilon^{\dagger}_{\sigma'}{}^{-1}\Big) - \Big(\Upsilon^{\dagger}_\sigma{}^{-1} \otimes \Upsilon^{\dagger}_{\sigma'}{}^{-1}\Big)r(\sigma - \sigma').$$

$$\{\Upsilon^{\dagger}_\sigma{}^{-1} \overset{\otimes}{,} \Upsilon_{\sigma'}\}^* = r(\sigma - \sigma')\Big(\Upsilon^{\dagger}_\sigma{}^{-1} \otimes \Upsilon'_\sigma\Big) - \Big(\Upsilon^{\dagger}_\sigma{}^{-1} \otimes \Upsilon'_\sigma\Big)\Big)(r + iC).$$

$$\{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}^*_{op} = \{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}^*; \quad \{\Upsilon^{\dagger}_\sigma{}^{-1} \overset{\otimes}{,} \Upsilon^{\dagger}_{\sigma'}{}^{-1}\}^*_{op} = \{\Upsilon^{\dagger}_\sigma{}^{-1} \overset{\otimes}{,} \Upsilon^{\dagger}_{\sigma'}{}^{-1}\}^*; \tag{4.12}$$

$$\{\Upsilon_\sigma^{\dagger\,-1} \overset{\otimes}{,} \Upsilon_{\sigma'}'\}_{op}^* = (r+iC)\Big(\Upsilon_\sigma^{\dagger\,-1} \otimes \Upsilon_\sigma'\Big) - \Big(\Upsilon_\sigma^{\dagger\,-1} \otimes \Upsilon_\sigma'\Big)r(\sigma-\sigma').$$

Here the canonical r-matrix and the Casimir element C are given by the standard expressions

$$r = \sum_{\alpha>0} \frac{i|\alpha|^2}{2}(E^{-\alpha} \otimes E^\alpha - E^\alpha \otimes E^{-\alpha}),$$

$$C = \sum_\mu H^\mu \otimes H^\mu + \sum_{\alpha>0} \frac{|\alpha|^2}{2}(E^{-\alpha} \otimes E^\alpha + E^\alpha \otimes E^{-\alpha}),$$

moreover, we set

$$r(\sigma-\sigma') = i\sum_\mu (H^\mu \otimes H^\mu)(1 + 2\sum_{n>0} e^{in(\sigma-\sigma')}) +$$

$$+i\sum_{\alpha>0} |\alpha|^2 (E^{-\alpha}\otimes E^\alpha) + i\sum_\alpha |\alpha|^2 (E^{-\alpha}\otimes E^\alpha)\sum_{n>0} e^{in(\sigma-\sigma')} = r + C\,\text{cotg}\frac{\sigma-\sigma'}{2}.$$

$$(4.13)$$

For completeness, we evaluate explicitely also the left and the right Poisson-Lie brackets $\{.,.\}_L^*$ and $\{.,.\}_R^*$ associated to the affine Poisson structure Π^*, by using the defining formulae (2.2) and (2.3) with all subscripts L, R set to L only, for $\{.,.\}_L^*$, and to R only, for $\{.,.\}_R^*$:

$$\{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}_L^* = \{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}_R^* = \{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}^*;$$

$$\{\Upsilon_\sigma^{\dagger\,-1} \overset{\otimes}{,} \Upsilon_{\sigma'}^{\dagger\,-1}\}_L^* = \{\Upsilon_\sigma^{\dagger\,-1} \overset{\otimes}{,} \Upsilon_{\sigma'}^{\dagger\,-1}\}_R^* = \{\Upsilon_\sigma^{\dagger\,-1} \overset{\otimes}{,} \Upsilon_{\sigma'}^{\dagger\,-1}\}^*;$$

$$\{\Upsilon_\sigma^{\dagger\,-1} \overset{\otimes}{,} \Upsilon_{\sigma'}'\}_L^* = r(\sigma-\sigma')\Big(\Upsilon_\sigma^{\dagger\,-1} \otimes \Upsilon_\sigma'\Big) - \Big(\Upsilon_\sigma^{\dagger\,-1} \otimes \Upsilon_\sigma'\Big)r(\sigma-\sigma').$$

$$\{\Upsilon_\sigma^{\dagger\,-1} \overset{\otimes}{,} \Upsilon_{\sigma'}'\}_R^* = (r+iC)\Big(\Upsilon_\sigma^{\dagger\,-1} \otimes \Upsilon_\sigma'\Big) - \Big(\Upsilon_\sigma^{\dagger\,-1} \otimes \Upsilon_\sigma'\Big)(r+iC).$$

Let us define the following Hermitian matrix valued observables on the phase space $S_\infty = G_R G^* \cap G^* G_L$ of the $q \to \infty$ WZW model:

$$L(\sigma) = \Lambda_L^*(\Upsilon_\sigma \Upsilon_\sigma^\dagger), \quad R(\sigma) = \Lambda_R^*(\Upsilon_\sigma^\dagger \Upsilon_\sigma).$$

By using the Poisson properties of the maps $\Lambda_L : (S_\infty, \Pi_D^\infty) \to (G^*, \Pi_{op}^*)$ and $\Lambda_R : (S_\infty, \Pi_D^\infty) \to (G^*, \Pi^*)$, we find out the basic commutation relations of the left and the right ∞-current algebras:

$$\{L(\sigma) \overset{\otimes}{,} L(\sigma')\}_D^\infty =$$

18

$$= \left(L(\sigma)\otimes L(\sigma')\right)\left(r+C\mathrm{cotg}\frac{\sigma-\sigma'}{2}\right)+\left(r+C\mathrm{cotg}\frac{\sigma-\sigma'}{2}\right)\left(L(\sigma)\otimes L(\sigma')\right)$$

$$-\left(L(\sigma)\otimes 1\right)\left(r+iC\right)\left(1\otimes L(\sigma')\right)-\left(1\otimes L(\sigma')\right)\left(r-iC\right)\left(L(\sigma)\otimes 1\right), \quad (4.14)$$

$$\{R(\sigma)\overset{\otimes}{,}R(\sigma')\}_D^\infty =$$

$$= -\left(R(\sigma)\otimes R(\sigma')\right)\left(r+C\mathrm{cotg}\frac{\sigma-\sigma'}{2}\right)-\left(r+C\mathrm{cotg}\frac{\sigma-\sigma'}{2}\right)\left(R(\sigma)\otimes R(\sigma')\right)+$$

$$+\left(R(\sigma)\otimes 1\right)\left(r-iC\right)\left(1\otimes R(\sigma')\right)+\left(1\otimes R(\sigma')\right)\left(r+iC\right)\left(R(\sigma)\otimes 1\right). \quad (4.15)$$

The Poisson brackets (4.14) and (4.15) are the $q\to\infty$ analogues of the ordinary $q\to 1$ left and right Kac-Moody relations (2.66) and (2.67) of Ref. [9]. Actually, for a generic finite q, we have concentrated ourselves in [9] mainly on the left chiral version of the q-WZW model therefore we have detailed in (1.37) of [9] only the left q-Kac-Moody brackets. It is in fact the bracket (1.37) of [9] which is the finite q analogue of our left ∞-bracket (4.33).

4.3   Chiral decomposition

Both the standard WZW model [21, 8, 2] and its q-deformation [9] admit the so called chiral decomposition. This means, roughly speaking, that the phase space S_q of the model for each finite q can be represented as a "square" of a simpler symplectic manifold M_q which itself enjoys only one half of the full symmetry of S_q. More precisely, by the "square" of a symplectic manifold M_q we mean the symplectic manifold $S_q = M_q \times M_q // T$ where the notation $//T$ means the symplectic reduction by an appropriate action of the Cartan torus $T \subset K$ on $M_q \times M_q$. The aim of the section is to show that the chiral decomposition takes place also in the $q\to\infty$ case. However, there is a novelty: a remarkable duality in the description of the chiral symplectic manifold M_∞ related to the fact that the groups G_R and G_L are not isomorphic. We shall describe this duality in Sec 4.6 and, for the moment, we restrict ourselves to the explicite description of the chiral decomposition of the phase space $S_\infty = G_R G^* \cap G^* G_L = G_R G^*$. Let us start by formulating and proving an auxiliary theorem on ∞-Cartan decomposition. We remind that, in order to avoid cumbersome notations, we shall not write explicitely the injection

homomorphisms ς, hoping that the reader will restore them easily from the context.

Theorem 4 *For very element $s \in S_\infty$, there exist two elements $k_l, k_r \in G_R$ and an element $a \in A_+$ such that*

$$s = k_l a \Xi_R(k_r a). \tag{4.16}$$

The ambiguity of this decomposition is given by the simultaneous right multiplication $(k_l, k_r) \to (k_l t, k_r t)$ by any element t of the Cartan torus $T \subset K$.

Reciprocally, for every elements $k_l, k_r \in G_R$ and every element $a \in A_+$ the product $k_l a \Xi_R(k_r a)$ is in S_∞.

Proof: First of all we remind the notation. Thus A is the (real) subgroup of $K^{\mathbf{C}}$ given by the Iwasawa decomposition $K^{\mathbf{C}} = KAN$. Its Lie algebra $\mathcal{A}$ consists of the Hermitian elements of the Cartan subalgebra $\mathcal{H}^{\mathbf{C}}$ and it is spanned by the elements H^μ (cf. Sec 4.1). By A_+ we mean $\exp \mathcal{A}_+$ where $\mathcal{A}_+$ is the positive Weyl chamber in $\mathcal{A}$. For instance, for the groups $K^{\mathbf{C}} = SL(n, \mathbf{C})$, A_+ consists of diagonal matrices with real positive entries ordered from the biggest one to the smallest one. We remind also that the Lie algebra $\mathcal{T}$ of the Cartan torus T is spanned by the elements $T^\mu = iH^\mu$. Finally, we recall that the decomposition $D = G^* G_L$ is global hence the domain of definition of the map Ξ_R (introduced in Eq. (3.1)), is the whole double D.

The (hermitian conjugated version of the) theorem 8.1.1. of [18] says that every element $u \in L_{pol}K^{\mathbf{C}}$ can be decomposed as $u = u_- u_0$ where the Fourier expansion (4.2) of u_- contains only the non-positive modes and u_0 is in the subgroup of $L_{pol}K$ consisting of the loops passing through the unit element of K at $\sigma = 0$. Obviously, u can be decomposed also as $u = u_N u' u_0$. Here u_0 is as before, u' is in $K^{\mathbf{C}}$ (in this context $K^{\mathbf{C}}$ is viewed as the subgroup of $LK^{\mathbf{C}}$ formed by the constant loops) and the Fourier expansion u_N contains as before only the non-positive modes with the zero mode being equal to the unit element of $K^{\mathbf{C}}$. By the classical theorem about the Cartan decomposition of a simple complex connected and simply connected group $K^{\mathbf{C}}$ (cf. [23], p. 117), we may write $u' = u_l a u_r^{-1}$, where u_l, u_r are the elements of K and a is in A_+. The ambiguity of this classical Cartan decomposition is given by the simultaneous right multiplication $(u_l, u_r) \to (u_l t, u_r t)$ by any element t

20

of the Cartan torus T. Thus we may decompose u also as

$$u = u_N u_l a u_r^{-1} u_0.$$

We note that $u_N u_l$ is an element of the subgroup G_R of $L_{pol} K^{\mathbf{C}}$ defined at the end of Sec 4.1 and $u_r^{-1} u_0$ is the element of the subgroup $G_L = L_{pol} K$. For the moment, we conclude that any u in $L_{pol} K^{\mathbf{C}}$ can be decomposed as

$$u = k_l a g_r^{-1},$$

where $k_l \in G_R$, $a \in A_+$ and $g_r \in G_L$. The ambiguity of this decomposition is given by the simultaneous right multiplication $(k_l, g_r) \rightarrow (k_l t, g_r t)$ by any element t of the Cartan torus T.

Suppose now, that the element u is in $S_\infty = G_R G^*$. This means that the element $a g_r^{-1}$ is in the domain of definition of the maps Ξ_L and Λ_R (cf. Eq. (3.1)). Set

$$k_r = \left(\Xi_L(a g_r^{-1}) \right)^{-1} \equiv \Xi_L^{-1}(a g_r^{-1}).$$

We see that k_r is an element of G_R. Since the domain of definition of the map Ξ_R is the whole double D, we can evaluate $\Xi_R(k_r a) \in G_L$:

$$\Xi_R(k_r a) = \Xi_R(\Xi_L^{-1}(a g_r^{-1}) a) = \Xi_R(\Lambda_R(a g_r^{-1}) \Xi_L^{-1}(a g_r^{-1}) a) = \Xi_R(g_r a^{-1} a) = g_r^{-1}.$$

We conclude that for every $u \in S_\infty$ there exist $k_l, k_r \in G_R$ and $a \in A_+$ such that

$$u = k_l a \Xi_R(k_r a).$$

It remains to deal with the ambiguity of this decomposition. When g_r is replaced by $g_r t$, $t \in T$ then k_r is replaced by $\Xi_L^{-1}(t^{-1} a g_r^{-1}) = \Xi_L^{-1}(a g_r^{-1}) t = k_r t$. This proves the first part of the theorem.

Reciprocally, let k_l, k_r be in G_R and $a \in A_+$. In order to show that $k_l a \Xi_R(k_r a)$ is in $S_\infty = G_R G^*$, it is obviously sufficient to show that $a \Xi_R(k_r a)$ is in S_∞. We use the fact that the maps Ξ_R and Λ_L are defined everywhere on $D = L_{pol} K^{\mathbf{C}}$ and we write

$$a \Xi_R(k_r a) = a \Xi_R(k_r a) \Lambda_L^{-1}(k_r a) \Lambda_L(k_r a) = a(k_r a)^{-1} \Lambda_L(k_r a) = k_r^{-1} \Lambda_L(k_r a).$$

The element $k_r^{-1} \Lambda_L(k_r a)$ is evidently in $G_R G^*$.

We have just established that the phase space S_∞ can be identified with the manifold $(G_R \times A_+ \times G_R)/T$. The core of this section is the following theorem expressing the chiral decomposability of the $q \to \infty$ WZW model:

Theorem 5 Parametrize by $k_l \in G_R$, $a_l \in A_+$ the direct product $M_\infty \equiv G_R \times A_+$ and define the following 2-form Ω_∞ on M_∞:

$$\Omega_\infty(k_l, a_l) = \frac{1}{2}(da_l a_l^{-1} \stackrel{\wedge}{,} k_l^{-1} dk_l)_\mathcal{D} + \frac{1}{2}(d\Xi_R(k_l a_l)\Xi_R^{-1}(k_l a_l) \stackrel{\wedge}{,} a_l^{-1} da_l + a_l^{-1}(k_l^{-1} dk_l)a_l)_\mathcal{D}.$$
(4.17)

Denote by $\phi : G_R \times A_+ \times G_R \to S_\infty$ the map induced by the ∞-Cartan decomposition, i.e.

$$\phi(k_l, a, k_r) = k_l a \Xi_R(k_r a). \tag{4.18}$$

Then the pull-back $\phi^* \omega_{S_\infty}$ of the grupoid symplectic form (3.6) can be written as

$$\phi^* \omega_{S_\infty} = \Omega_\infty(k_l, a_l = a) - \Omega_\infty(k_r, a_r = a).$$

Proof: The theorem says, in other words, that the restriction of the form $\Omega_\infty^l - \Omega_\infty^r$ on the submanifold of $M_\infty \times M_\infty$ defined by $a_l = a_r$ is the same thing as the ϕ^*-pull-back of the anomalous Semenov-Tian-Shansky form from S_∞ into $G_R \times A_+ \times G_R$. In order to prove it let s denote an element of S_∞. The grupoid symplectic form ω_{S_∞} is given by Eq.(3.6)

$$\omega_{P_\infty} = \frac{1}{2}(d\Lambda_L(s)\Lambda_L^{-1}(s) \stackrel{\wedge}{,} d\Xi_L(s)\Xi_L^{-1}(s))_\mathcal{D} + \frac{1}{2}(d\Lambda_R(s)\Lambda_R^{-1}(s) \stackrel{\wedge}{,} d\Xi_R(s)\Xi_R^{-1}(s))_\mathcal{D}.$$
(4.19)

By using the ∞-Cartan parametrization (4.16) and the relations (3.1), we easily infer:

$$\Lambda_L(s) = \Lambda_L(k_l a) = k_l a \Xi_R(k_l a), \tag{4.20}$$

$$\Lambda_R(s) = \Lambda_R(a\Xi_R(k_r a)) = \Xi_R^{-1}(k_r a)a^{-1}k_r^{-1}, \tag{4.21}$$

$$\Xi_L(s) = k_l \Xi_L(a\Xi_R(k_r a)) = k_l k_r^{-1}, \tag{4.22}$$

$$\Xi_R(s) = \Xi_R^{-1}(k_r a)\Xi_R(k_l a). \tag{4.23}$$

By inserting the expressions (4.20-23) into (4.19), we find immediately

$$\phi^* \omega_{S_\infty} = \frac{1}{2}(daa^{-1} \stackrel{\wedge}{,} k_l^{-1} dk_l)_\mathcal{D} + \frac{1}{2}(d\Xi_R(k_l a)\Xi_R^{-1}(k_l a) \stackrel{\wedge}{,} a^{-1} da + a^{-1}(k_l^{-1} dk_l)a)_\mathcal{D}$$

$$-\frac{1}{2}(daa^{-1} \overset{\wedge}{,} k_r^{-1}dk_r)_\mathcal{D} - \frac{1}{2}(d\Xi_R(k_r a)\Xi_R^{-1}(k_r a) \overset{\wedge}{,} a^{-1}da + a^{-1}(k_r^{-1}dk_r)a)_\mathcal{D} =$$

$$= \Omega_\infty(k_l, a_l = a) - \Omega_\infty(k_r, a_r = a).$$

#

Corollary *The chiral form Ω_∞ is symplectic. The manifold S_∞ can be obtained by the symplectic reduction $S_\infty = M_\infty \times M_\infty//T$, where T is the Cartan torus acting as $(k_l, k_r) \rightarrow (k_l t, k_r t)$, $t \in T$.*

Proof: The closedness of Ω_∞ can be seen from the fact that $\Omega_\infty(k_l, a_l)$ is the pull-back of the closed form ω_{S_∞} under the map $\chi : G_R \times A_+ \rightarrow S_\infty$ given by $\chi(k_l, a_l) = k_l a_l$. It is slightly more involved to show the non-degeneracy of Ω_∞ . First of all we compute the contraction $\iota_v \Omega_\infty$ where $v = L_{k_l *}T^\mu$. The result is

$$\Omega_\infty(., L_{k_l *}T^\mu) = (T^\mu, da_l a_l^{-1})_\mathcal{D} = d\psi_l^\mu, \tag{4.24}$$

where we have parametrized a_l as $a_l = e^{\psi_l^\mu H^\mu}$. We observe that, whatever is the point (k_l, a_l) in M_∞, the vector $L_{k_l *}T^\mu$ does not constitute a degeneracy direction of the form Ω_∞. If the form Ω_∞ had at some point a degeneracy vector, then the restriction of the form $\Omega_\infty(k_l, a_l) - \Omega_\infty(k_r, a_r)$ to the submanifold $a_l = a_r$ would have degeneracy vectors *other* than $L_{k_l *}T^\mu + L_{k_l *}T^\mu$. However, this is impossible since $\Omega_\infty(k_l, a_l) - \Omega_\infty(k_r, a_r)$ is the pull-back of the non-degenerate form ω_{P_∞}.

Having established that $(M_\infty, \Omega_\infty)$ is a symplectic manifold (actually it is going to play the role of the phase space of the chiral $q \rightarrow \infty$ WZW model), it is easy to prove that $S_\infty = M_\infty \times M_\infty//T$. Indeed, due to relative minus sign of the left and the right chiral symplectic forms on each copy of M_∞, we see from (4.24) that the Hamiltonian function $\psi_l^\mu - \psi_r^\mu$ generates the action of the vector $L_{k_l *}T^\mu + L_{k_r *}T^\mu$ on $M_\infty \times M_\infty$. The setting $a_l = a_r$ is nothing but saying that $\psi_l^\mu - \psi_r^\mu = 0$ and we conclude that the anomalous Semenov-Tian-Shansky form ω_{S_∞} comes indeed from the symplectic reduction $M_\infty \times M_\infty//T$.

#

Remark. The proof of the corollary is not rigorous by strictly mathematical standards and shifts the paper to the level of rigour common in mathematical physics. Indeed, the propositions in Sections 2 and 3 were proved only for finite dimensional Lie groups, in particular the result about the non-degeneracy of the grupoid symplectic form (3.6). In our study of the infinite-

23

dimensional loop groups, we still preserve the full mathematical rigour for certain propositions, e.g. Theorems 4 and 5, but in some cases we adopt the usual cavalier approach of mathematical physicists in treating the infinitesimal symplectic geometry of field theories. Thus, in the proof of the corollary, we assumed that the theorems of Sections 2 and 3 remains valid also in the infinite dimensional context.

4.4 Symmetry of the chiral model

There is a natural right action of the group G_R on the chiral phase space M_∞ given by $h \triangleright (k_l, a_l) = (h^{-1}k_l, a_l)$, $h \in G_R, (k_l, a_l) \in M_\infty$. It can be interpreted as the restriction to $M_\infty \subset S_\infty$ of the Poisson-Lie symmetric right action $G_R \times S_\infty \to S_\infty$ given by the group multiplication on the anomalous double: $h^{-1} \triangleright s = hs$, $h \in G_R, s \in S_\infty$. Our next goal is to prove that also the restricted (or chiral) G_R-action is in fact a Poisson-Lie symmetry.

Theorem 6 Denote Π^∞ the bivector inverse to the chiral symplectic form Ω_∞ and consider the map $\chi : G_R \times A_+ \to S_\infty$ given by $\chi(k_l, a_l) = k_l a_l$. Then it holds that the composition map $\Lambda_L \circ \chi : (M_\infty, \Pi^\infty) \to (G^*, \Pi^*_{op})$ is Poisson.

Proof: Consider a pair of functions x, y defined on G^*, their pull-backs $\Lambda_L^* x, \Lambda_L^* y$ defined on S_∞ and $\chi^* \Lambda_L^* x$, $\chi^* \Lambda_L^* y$ defined on M_∞, and also the functions $\chi^* \Lambda_L^* x \otimes 1, \chi^* \Lambda_L^* y \otimes 1$ defined on $M_\infty \times M_\infty$. Consider the submanifold O of $M_\infty \times M_\infty$ defined by setting $a_l = a_r$ and also the map $\phi : O \to S_\infty$ defined by Eq. (4.18). Then the ∞-Cartan parametrization (4.16) and the definition (3.1) of the map Λ_L imply that

$$(\chi^* \Lambda_L^* x \otimes 1)|_O = \phi^* \Lambda_L^* x. \tag{4.25}$$

Stated in words, the restriction of $\chi^* \Lambda_L^* x \otimes 1$ to O is the same thing as the ϕ^*-pull-back of the function $\Lambda_L^* x$ defined on S_∞. Moreover, we have

$$\Lambda_L(k_l t a_l) = \Lambda_L(k_l a_l t) = \Lambda_L(k_l a_l), \quad k_l \in G_R, a_l \in A_+, t \in T.$$

This relation implies that the function $\chi^* \Lambda_L^* x \otimes 1$ is T-invariant with respect to the Cartan torus action defined in the Corollary of Theorem 5. Since we know that $S_\infty = (M_\infty \times M_\infty)//T$, we infer

$$\{\chi^* \Lambda_L^* x \otimes 1, \chi^* \Lambda_L^* y \otimes 1\}_{M_\infty \times M_\infty}|_O = \phi^* \{\Lambda_L^* x, \Lambda_L^* y\}_D^\infty,$$

24

or, equivalently,

$$(\{\chi^*\Lambda_L^* x, \chi^*\Lambda_L^* y\}_{M_\infty} \otimes 1)|_O = \phi^*\{\Lambda_L^* x, \Lambda_L^* y\}_D^\infty. \qquad (4.26)$$

If we write the identity (4.25) for the function $\{x, y\}_{op}^*$, Theorem 1 combined with Eq. (4.26) gives

$$(\{\chi^*\Lambda_L^* x, \chi^*\Lambda_L^* y\}_{M_\infty} \otimes 1)|_O = (\chi^*\Lambda_L^*\{x, y\}_{op}^* \otimes 1)|_O.$$

Thus we conclude

$$\{\chi^*\Lambda_L^* x, \chi^*\Lambda_L^* y\}_{M_\infty} = \chi^*\Lambda_L^*\{x, y\}_{op}^*.$$

$$\#$$

Corollary *The right action of the group G_R on the ∞-WZW chiral phase space M_∞, given by $h \triangleright (k, a) = (h^{-1}k, a)$, $h \in G_R$, is the right Poisson-Lie symmetry corresponding to the moment map $\Lambda_L \circ \chi$.*

Proof: Consider a point (k, a) in M_∞. The multiplication of (k, a) on the left by an infinitesimal generator $S \in \mathcal{G}_R$ gives the vector $v_S = (Sk, a)$ and its χ_*-push-forward vector $\chi_* v_S = Ska = R_{(ka)*}S \in T_{(ka)}S_\infty$. Denote also by v_S and $\chi_* v_S$ the respective vector fields on M_∞ and S_∞ obtained by varying the point (k, a). In the sense of Lemma 5, Theorem 1 and Eq. (3.4) of Sec 4.3, we know that

$$-\chi_* v_S = \Pi_D^\infty(\Lambda_L^*(\rho, S)_\mathcal{D}, .).$$

This relation can be rewritten equivalently as

$$\omega_{S_\infty}(., \chi_* v_S) = \Lambda_L^*(\rho, S)_\mathcal{D} \qquad (4.27)$$

(note that the bivector inverse to the groupoid symplectic form ω_{S_∞} on S_∞ has been denoted by Π_D^∞). The χ^*-pull-back of Eq. (4.27) gives

$$\Omega_\infty(., v_S) = \chi^*\Lambda_L^*(\rho, S)_\mathcal{D}, \qquad (4.28)$$

or, equivalently,

$$v_S = -\Pi^\infty((\Lambda_L \circ \chi)^*(\rho, S)_\mathcal{D}, .). \qquad (4.29)$$

$$\#$$

4.5 Exchange relations

Our next goal is to invert the chiral symplectic form Ω_∞ on $M_\infty = G_R \times A_+$. The strategy, which we shall use, will be that of Sec 5.1.2 of [9]. It consists in exploiting particular properties of the Lie derivatives of Poisson-Lie symmetric bivectors. Let u be a differential 1-form on M_∞ and $v = R_* S$, $S \in \mathcal{G}_R$ a right-invariant vector field on G_R viewed as the vector field on M_∞. We calculate the Lie derivate $\mathcal{L}_v$ of the both sides of the following identity

$$\Omega_\infty(., \Pi^\infty(., u)) = u$$

and we obtain

$$(\mathcal{L}_v \Omega_\infty)(., \Pi^\infty(., u)) + \Omega_\infty(., (\mathcal{L}_v \Pi^\infty)(., u)) = 0.$$

From Eq. (4.28), we infer

$$\mathcal{L}_v \Omega_\infty = d(\iota_v \Omega_\infty) = -d(\chi^* \Lambda_L^*(\rho, S)_\mathcal{D})) = -\chi^* \Lambda_L^*((\rho, S')_\mathcal{D} \wedge (\rho, S'')_\mathcal{D}), \quad (4.30)$$

where the map $S \to S' \wedge S'' \in \mathcal{G}_R \wedge \mathcal{G}_R$ is the dual to the Lie algebra commutator $[.,.]^* : \mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^*$. The last equality follows from the well-known Maurer-Cartan identity

$$d(R^* S) = R^* S' \wedge R^* S'', \quad S \in (\mathcal{G}^*)^*, \quad (4.31)$$

because $\mathcal{G}_R$ can be identified with $(\mathcal{G}^*)^*$. Finally, by inserting (4.30) into (4.31), we arrive at the following formula

$$\mathcal{L}_v \Pi^\infty = -v' \wedge v",$$

where $v' \equiv R_* S'$ and $v'' \equiv R_* S''$.

With the notations introduced after Lemma 3, introduce the following Poisson-Lie bivector Π_R on the group manifold G_R:

$$R_{g^{-1}*} \Pi_R(g)(t_1, t_2) \equiv -(Ad_{\varsigma_R(g^{-1})} \varsigma(t_1), p_R^* Ad_{\varsigma_R(g^{-1})} \varsigma(t_2))_\mathcal{D}, \quad g \in G_R, t_1, t_2 \in \mathcal{G}^* \quad (4.32)$$

and consider it as the bivector on M_∞. Then a straightforward calculation gives

$$\mathcal{L}_v(\Pi^\infty + \Pi_R) = 0,$$

26

hence $\Pi^\infty + \Pi_R$ must have the following shape

$$\Pi^\infty + \Pi_R = \Sigma_{ij}(a)L_*(T_R^i \wedge T_R^j) + \sigma_i^\mu(a)L_*T_R^i \wedge \frac{\partial}{\partial\phi^\mu} + s^{\mu\nu}(a)\frac{\partial}{\partial\phi^\mu} \wedge \frac{\partial}{\partial\phi^\nu},$$

where $a = \exp\left(\sum \phi^\mu H^\mu\right)$ and T_R^i is a basis in $\mathcal{G}_R$. The crucial observation is as follows: since at the group unit $e_R \in G_R$ the Poisson-Lie bivector Π_R vanishes, in order to fully determine the unknown functions $\Sigma_{ij}(a), \sigma_i^\mu(a)$ and $s^{\mu\nu}(a)$, it is sufficient to invert the chiral symplectic form Ω_∞ just at the points (e_R, a).

We may parametrize a small vicinity of the group origin e_R by the coordinates on the Lie algebra $\mathcal{G}_R$ corresponding to the basis $(T^\mu, B_R^{\hat{\alpha}}, C_R^{\hat{\alpha}}), \hat{\alpha} > 0$ introduced in Sec 4.1. Thus we parametrize any element $\zeta \in \mathcal{G}_R$ as

$$\zeta = \tau_\mu T^\mu + \beta_{\hat{\alpha}} B_R^{\hat{\alpha}} + \gamma_{\hat{\alpha}} C_R^{\hat{\alpha}}$$

and find the following expression for the symplectic form Ω_∞ in (e_R, a)

$$\Omega_\infty(e_{G_R}, a) = d\phi^\mu \wedge d\tau_\mu + \sum_{\hat{\alpha}>0, n>0} \frac{a^{2\alpha}}{|\hat{\alpha}|^2}d\beta_{\hat{\alpha}} \wedge d\gamma_{\hat{\alpha}} + \sum_{\hat{\alpha}>0, n=0} \frac{a^{2\alpha}-1}{|\hat{\alpha}|^2}d\beta_{\hat{\alpha}} \wedge d\gamma_{\hat{\alpha}}.$$

$$(4.33)$$

Here we use the notation
$$a^{2\alpha} \equiv e^{2\alpha(\phi^\mu H^\mu)}.$$

It is easy to invert this almost Darboux-like expression (4.33) and to write (everywhere in M_∞):

$$\Pi^\infty = -\Pi_{G_R} + L_*T^\mu \wedge \frac{\partial}{\partial\phi^\mu} - \sum_{\hat{\alpha}>0, n>0} a^{-2\alpha}|\hat{\alpha}|^2 L_*(B_R^{\hat{\alpha}} \wedge C_R^{\hat{\alpha}}) - \sum_{\hat{\alpha}>0, n=0} \frac{|\hat{\alpha}|^2}{a^{2\alpha}-1}L_*(B_R^{\hat{\alpha}} \wedge C_R^{\hat{\alpha}}).$$

$$(4.34)$$

We wish to calculate the brackets of the principal variables $a \in A_+$ and $k \in G_R$ like $\{a \overset{\otimes}{,} k(\sigma)\}_\infty$, $\{k(\sigma) \overset{\otimes}{,} k(\sigma')\}_\infty$, $\{k(\sigma) \overset{\otimes}{,} k(\sigma')^{\dagger^{-1}}\}_\infty$, etc. or more precisely, the brackets $\{a \overset{\otimes}{,} \Upsilon_\sigma\}_\infty$, $\{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}\}_\infty$, $\{\Upsilon_\sigma \overset{\otimes}{,} \Upsilon_{\sigma'}^{\dagger^{-1}}\}_\infty$ etc. where the matrix valued functions Υ_σ (introduced in Sec 4.2) are restricted to the subgroup $G_R \subset L_{pol}K^{\mathbf{C}}$. However, we prefer to make explicit slightly modified Poisson brackets like $\{a \overset{\otimes}{,} k(\sigma)a\}_\infty$, $\{k(\sigma)a \overset{\otimes}{,} k(\sigma')a\}_\infty$, $\{k(\sigma) \overset{\otimes}{,} (k(\sigma')a)^{\dagger^{-1}}\}_\infty$, since they contain equivalent information and are

less cumbersome. With the help of Eqs.(4.32),(4.34), (4.9) and (4.10), we find

$$\{a \overset{\otimes}{,} k(\sigma)a\}_\infty = -i(a \otimes k(\sigma)a)(H^\mu \otimes H^\mu), \qquad (4.35)$$

$$\{k(\sigma)a \overset{\otimes}{,} k(\sigma')a\}_\infty = (r + C\cotg\frac{\sigma - \sigma'}{2}) \times$$

$$\times \left(k(\sigma)a \otimes k(\sigma')a \right) - \left(k(\sigma)a \otimes k(\sigma')a \right)(r(a) + C\cotg\frac{\sigma - \sigma'}{2}), \qquad (4.36)$$

$$\{(k(\sigma)a)^{\dagger^{-1}} \overset{\otimes}{,} (k(\sigma')a)^{\dagger^{-1}}\}_\infty = (r + C\cotg\frac{\sigma - \sigma'}{2}) \times$$

$$\times \left((k(\sigma)a)^{\dagger^{-1}} \otimes (k(\sigma')a)^{\dagger^{-1}} \right) - \left((k(\sigma)a)^{\dagger^{-1}} \otimes (k(\sigma')a)^{\dagger^{-1}} \right)(r(a) + C\cotg\frac{\sigma - \sigma'}{2}),$$
$$(4.37)$$

$$\{k(\sigma)a \overset{\otimes}{,} (k(\sigma')a)^{\dagger^{-1}}\}_\infty =$$

$$= (r - iC)\left(k(\sigma)a \otimes (k(\sigma')a)^{\dagger^{-1}} \right) - \left(k(\sigma)a \otimes (k(\sigma')a)^{\dagger^{-1}} \right)(r(a) + C\cotg\frac{\sigma - \sigma'}{2}),$$
$$(4.38)$$

where

$$r(a) = \sum_\alpha \frac{i|\alpha|^2}{2} \frac{a^\alpha + a^{-\alpha}}{a^\alpha - a^{-\alpha}} E^{-\alpha} \otimes E^\alpha. \qquad (4.39)$$

Note that the summation in (4.39) runs over all roots α and not only over the positive roots. It is not difficult to recognize in the expression (4.39) the canonical dynamical r-matrix associated to a simple Lie algebra (cf. Eq.(5.60) of [9]).

The Poisson brackets (4.35-38) encode full information about the symplectic structure of the chiral ∞-WZW model and they may be called the exchange relations. However, it is not obvious how to compare them with the exchange relations of the finite q WZW model (cf. Eq.(5.159) of Ref.[9]). The point is that the our ∞-exchange relations involve the Poisson brackets of certain matrix-valued functions on the group G_R while the exchange relations for finite q involve the Poisson brackets of the matrix-valued functions on the group $G_L = L_{pol}K$. It turns out, however, that the chiral ∞-WZW model enjoys a remarkable duality which enables to describe its dynamics also in terms of the functions on the group G_L. Thus, in particular, we shall see in the next section that the exchange relations (4.35-38) can be equivalently rewritten in terms of dual exchange relations employing the matrix-valued

28

functions on G_L. It will turn out, rather satisfactorily, that the dual ∞-exchange relation can be obtained by the direct $q \to \infty$ limit of the finite q exchange relations.

4.6   Duality

We have learned in Sec 3, that the anomalous Poisson-Lie moment maps realize at the same time the right and the left Poisson-Lie symmetries. In Corollary of Theorem 6, we have worked out the action of the right infinitesimal $\mathcal{G}_R$ symmetry on the phase space M_∞ of the chiral ∞-WZW model and we found that it can be lifted to the natural global right action of the group G_R on itself. More precisely it is given by $h \triangleright (k, a) = (h^{-1}k, a)$, $h \in G_R$, $(k, a) \in G_R \times A_+ = M_\infty$. Consider now the infinitesimal action of the left $\mathcal{G}_L$ Poisson-Lie symmetry generated by the vector fields $\Pi^\infty(., (\Lambda_L \circ \chi)^*(\lambda, T)_\mathcal{D})$ for $T \in \mathcal{G}_L$ and we may ask the following question: Can we parametrize the points of the phase space M_∞ by using the group G_L in such a way that the action of the left $\mathcal{G}_L$ symmetry becomes just (the infinitesimal version of) the natural action of the group G_L on itself? It turns out that the answer to this question is affirmative, i.e. we shall succeed to represent M_∞ as a submanifold of the direct product $G_L \times A_-$ on which the Lie algebra $\mathcal{G}_L$ (but not the group G_L) acts by the right-invariant vector fields on G_L. The quantitative basis of this result is the folowing theorem:

Theorem 7 Parametrize by $\tilde{k} \in G_L$, $\tilde{a} \in A_-$ the direct product $G_L \times A_-$, where $A_- = \exp \mathcal{A}_-$ and $\mathcal{A}_- = -\mathcal{A}_+$ is the negative Weyl chambre. Define a 2-form $\tilde{\Omega}_\infty(\tilde{k}, \tilde{a})$ on the *submanifold* $\tilde{M}_\infty$ of $G_L \times A_-$ for which $\tilde{a}^{-1}\tilde{k}^{-1}$ is in the domain of definition of the maps Ξ_L and Λ_R (i.e. $\tilde{a}^{-1}\tilde{k}^{-1}$ is in S_∞):

$$\tilde{\Omega}_\infty(\tilde{k}, \tilde{a}) = \frac{1}{2}(d\tilde{a}\tilde{a}^{-1} \overset{\wedge}{,} \tilde{k}^{-1}d\tilde{k})_\mathcal{D} + \frac{1}{2}(d\Xi_L(\tilde{a}^{-1}\tilde{k}^{-1})\Xi_L^{-1}(\tilde{a}^{-1}\tilde{k}^{-1}) \overset{\wedge}{,} \tilde{a}^{-1}d\tilde{a} + \tilde{a}^{-1}(\tilde{k}^{-1}d\tilde{k})\tilde{a})_\mathcal{D}.$$
$$(4.40)$$

Then the manifold $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$ is symplectic and the infinitesimal version of the right G_L action $\tilde{h} \triangleright (\tilde{k}, \tilde{a}) = (\tilde{h}^{-1}\tilde{k}, \tilde{a})$, $\tilde{h} \in G_L$ is a right Poisson-Lie symmetry of $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$.

Proof: Consider a map $U : M_\infty \to \tilde{M}_\infty$ defined everywhere in M_∞ as follows

$$(\tilde{k}, \tilde{a}) = U(k, a) = (\Xi_R^{-1}(ka), a^{-1}), \quad (k, a) \in M_\infty. \qquad (4.41)$$

Similarly, consider a map $V : \tilde{M}_\infty \to M_\infty$ defined everywhere in $\tilde{M}_\infty$ as follows

$$(k, a) = V(\tilde{k}, \tilde{a}) = (\Xi_L^{-1}(\tilde{a}^{-1}\tilde{k}^{-1}), \tilde{a}^{-1}), \quad (\tilde{k}, \tilde{a}) \in \tilde{M}_\infty.$$

We easily verify that $U \circ V$ is the identity map on M_∞ and $V \circ U$ is the identity map on $\tilde{M}_\infty$. It follows that both U, V are injective and surjective, hence they are diffeomorphisms inverse to each other.

By an easy direct calculation, we can relate the form (4.40) on $\tilde{M}_\infty$ to the symplectic form (4.17) on M_∞ by pull-backs

$$U^* \tilde{\Omega}_\infty = -\Omega_\infty, \quad V^* \Omega_\infty = -\tilde{\Omega}_\infty. \tag{4.42}$$

This implies that $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$ is a symplectic manifold.

We pick $T \in \mathcal{G}_L$ and we push forward by U the vector field $\Pi(., \chi^* \Lambda_L^* < \lambda, T >) \in Vect(M_\infty)$, realizing the *left* $\mathcal{G}_L$ Poisson-Lie symmetry of M_∞. The result is a vector field $\tilde{w}_T \in Vect(\tilde{M}_\infty)$ given by

$$\tilde{w}_T \equiv U_*(\Pi^\infty(., \chi^* \Lambda_L^* < \lambda, T >)) = -\tilde{\Pi}^\infty(., V^* \chi^* \Lambda_L^* < \lambda, T >) = \tilde{\Pi}^\infty(., V^* \chi^* \Lambda_L^* J^* < \rho, T >),$$

where $\tilde{\Pi}^\infty$ stands for the Poisson bivector inverse to the symplectic form $\tilde{\Omega}_\infty$, the map $\chi : M_\infty \to S_\infty$ was defined in Theorem 6 and the map $J : G^* \to G^*$ is just the inversion map $J(b) = b^{-1}$, $b \in G^*$. Note that $J^* \rho = -\lambda$.

Now consider a map $\tilde{\chi} : \tilde{M}_\infty \to S_\infty$ given by $\tilde{\chi}(\tilde{k}, \tilde{a}) = \tilde{a}^{-1}\tilde{k}^{-1}$. Then it is easy to see that

$$\Lambda_R \circ \tilde{\chi} = J \circ \Lambda_L \circ \chi \circ V, \tag{4.43}$$

which permits to rewrite

$$\tilde{w}_T = \tilde{\Pi}^\infty(., \tilde{\chi}^* \Lambda_R^* < \rho, T >).$$

Obviously, we have

$$\tilde{\Omega}_\infty(., \tilde{w}_T) = \tilde{\chi}^* \Lambda_R^* (\rho, T)_\mathcal{D}. \tag{4.44}$$

Consider a point $(\tilde{k}, \tilde{a}) \in \tilde{M}_\infty$. The multiplication of $(\tilde{k}, \tilde{a})$ on the left by an infinitesimal generator $T \in \mathcal{G}_L$ gives the vector $\tilde{v}_T = (T\tilde{k}, \tilde{a})$ and its $\tilde{\chi}_*$-push-forward vector $\tilde{\chi}_* \tilde{v}_T = -\tilde{a}^{-1}\tilde{k}^{-1}T = -L_{(\tilde{a}^{-1}\tilde{k}^{-1})_*}T \in T_{(\tilde{a}^{-1}\tilde{k}^{-1})_*}S_\infty$. Denote also by $\tilde{v}_T$ and $\tilde{\chi}_* \tilde{v}_T$ the respective vector fields on $\tilde{M}_\infty$ and S_∞ obtained by varying the point $(\tilde{k}, \tilde{a})$. Then we learn from Theorem 2 (cf. Eq. (3.5)):

$$\tilde{\chi}_* \tilde{v}_T = \Pi_D^\infty(., \Lambda_R^* (\rho, T)_\mathcal{D}),$$

or, equivalently,

$$\omega_{S_\infty}(., \tilde{\chi}_* \tilde{v}_T) = \Lambda_R^*(\rho, T)_{\mathcal{D}}. \tag{4.45}$$

The relation (4.42) and Theorem 5 imply easily

$$\tilde{\Omega}_\infty = \tilde{\chi}^* \omega_{S_\infty},$$

where ω_{S_∞} is the grupoid symplectic form (3.6). This fact permits to infer from (4.45):

$$\tilde{\Omega}_\infty(., \tilde{v}_T) = \tilde{\chi}^* \Lambda_R^*(\rho, T)_{\mathcal{D}}. \tag{4.46}$$

By comparing Eqs. (4.44) and (4.46), we obtain

$$\tilde{w}_T = \tilde{v}_T. \tag{4.47}$$

Thus we have established, that the *left* $\mathcal{G}_L$ Poisson-Lie symmetry of the chiral phase space M_∞ is most conveniently described by using the (anti) - symplectomorphism $U : M_\infty \to \tilde{M}_\infty$. Indeed, as Eq. (4.47) shows, in the parametrization $\tilde{M}_\infty = \tilde{G}_L \times A_-$ the $\mathcal{G}_L$ symmetry action is nothing but the infinitesimal version of the natural *left* G_L action $\tilde{h} \triangleright (\tilde{k}, \tilde{a}) = (\tilde{h}\tilde{k}, \tilde{a})$, $\tilde{h} \in G_L$.

Now the proof of the present theorem is almost finished. Obviously, the *right* action of G_L on $\tilde{M}_\infty$, given by $\tilde{h} \triangleright (\tilde{k}, \tilde{a}) = (\tilde{h}^{-1}\tilde{k}, \tilde{a})$, $\tilde{h} \in G_L$, is infinitesimaly generated by the vector fields $-\tilde{v}_T$, $T \in \mathcal{G}_L$. We infer from (4.44) and (4.47) that

$$-\tilde{v}_T = \tilde{\Pi}^\infty(\tilde{\chi}^* \Lambda_R^* < \rho, T >, .) = \tilde{\Pi}^\infty((\Lambda_R \circ \tilde{\chi})^* < \rho, T >, .)$$

It remains to show that the map $(\Lambda_R \circ \tilde{\chi}) : (\tilde{M}_\infty, \tilde{P}_\infty) \to (G^*, \Pi^*)$ is Poisson, thus realizing the Poisson-Lie symmetry of $\tilde{M}_\infty$. The Poisson property of the map $\Lambda_R \circ \tilde{\chi}$ follows from Eq. (4.43). Indeed, we know from (4.42) that $V : (\tilde{M}_\infty, \tilde{\Pi}^\infty) \to (M_\infty, -\Pi^\infty)$ is the Poisson map, Theorem 6 states that $\Lambda_L \circ \chi : (M_\infty, \Pi^\infty) \to (G^*, \Pi^*_{op})$ is the Poisson map and, by using Eqs. (2.2),(2.3), it is not difficult to work out that $J : (G^*, \Pi^*_{op}) \to (G^*, -\Pi^*)$ is the Poisson map. Thus the composition map $J \circ \Lambda_L \circ \chi \circ V = \Lambda_R \circ \tilde{\chi}$ is also Poisson.
$$\#$$

Interpretation in terms of duality. It is not so surprising that there is an (anti)-symplectomorphism U transforming the infinitesimal *left* $\mathcal{G}_L$ Poisson-Lie symmetry of M_∞ into a more nicely looking *right* $\mathcal{G}_L$ Poisson-Lie symmetry of $\tilde{M}_\infty$. After all, to find this anti-symplectomorphism, it was sufficient to integrate the vector fields $\Pi^\infty(., (\Lambda_L \circ \chi)^*(\lambda, T)_{\mathcal{D}})$ to G_L-orbits and use the

31

G_L orbits for the parametrization of the chiral phase space M_∞ (it is in this way that we have constructed the diffeomorphism U). What is surprising and remarkable is that the U-transformed expression (4.40) has *the same structure* than the original one (4.17). Indeed, we have

$$\Omega_\infty(k,a) = \frac{1}{2}(daa^{-1} \overset{\wedge}{,} k^{-1}dk)_\mathcal{D} + \frac{1}{2}((\Xi_R \circ \chi)^* \rho_R \overset{\wedge}{,} a^{-1}da + a^{-1}(k^{-1}dk)a)_\mathcal{D},$$

$$\tilde{\Omega}_\infty(\tilde{k},\tilde{a}) = \frac{1}{2}(d\tilde{a}\tilde{a}^{-1} \overset{\wedge}{,} \tilde{k}^{-1}d\tilde{k})_\mathcal{D} + \frac{1}{2}((\Xi_L \circ \tilde{\chi})^* \rho_L \overset{\wedge}{,} \tilde{a}^{-1}d\tilde{a} + \tilde{a}^{-1}(\tilde{k}^{-1}d\tilde{k})\tilde{a})_\mathcal{D},$$

where ρ_R and ρ_L are, respectively, the right-invariant Maurer-Cartan forms on G_R and G_L. We interpret this phenomenon as the $\mathcal{G}_L \leftrightarrow \mathcal{G}_R$ duality of the chiral $q \to \infty$ WZW model. This duality relates the Poisson-Lie symmetries of $(M_\infty, \Omega_\infty)$ and $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$ in a nice way: we have shown that the left $\mathcal{G}_L$ symmetry of $(M_\infty, \Omega_\infty)$ gets U-transformed into the right $\mathcal{G}_L$ symmetry of $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$ and it is not difficult to show that the right $\mathcal{G}_R$ symmetry of $(M_\infty, \Omega_\infty)$ gets U-transformed into the left $\mathcal{G}_R$ symmetry of $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$.

Remark. Corollary of Theorem 6 tells us that the right $\mathcal{G}_R$ Poisson-Lie symmetry of M_∞ is given by the infinitesimal version of the natural global right G_R action on $M_\infty = G_R \times A_+$, hence we see that the infinitesimal right $\mathcal{G}_R$ symmetry can be lifted to the *global* G_R symmetry. However, the left $\mathcal{G}_L$ Poisson-Lie symmetry of M_∞ *cannot* be lifted to a global action of the group $G_L = L_{pol}K$ on M_∞. This becomes evident upon the U-transformation of M_∞ into $\tilde{M}_\infty$. Indeed, the manifold $\tilde{M}_\infty$ is the *proper* subset of the direct product $G_L \times A_-$ and the global action of the group G_L on itself does not respect this proper subset. Thus we observe, that a moment map can realize at the same time a global right symmetry and a local left symmetry. In this sense, the duality $\mathcal{G}_L \leftrightarrow \mathcal{G}_R$ is only local and it cannot be lifted to a global $G_L \leftrightarrow G_R$ duality.

4.7 $q \to \infty$ limit of the exchange relations

Now it is time to work out the dual exchange relations, i.e. to calculate the Poisson brackets of the type $\{\tilde{a}, \tilde{k}(\sigma)\}_\infty$, $\{\tilde{k}(\sigma) \overset{\otimes}{,} \tilde{k}(\sigma')\}_\infty$. Actually, those two brackets characterize completely the dual symplectic structure $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$. It is because $\tilde{k}$ is the element of $G_L = L_{pol}K$ which means that $\tilde{k}(\sigma)^{\dagger^{-1}} = \tilde{k}(\sigma)$ and it is not necessary to calculate the brackets $\{\tilde{k}(\sigma) \overset{\otimes}{,}$

$\tilde{k}(\sigma')^{\dagger^{-1}}\}_\infty$, $\{\tilde{k}(\sigma)^{\dagger^{-1}} \overset{\otimes}{,} \tilde{k}(\sigma')^{\dagger^{-1}}\}_\infty$ like in the G_R case. Our strategy for working out the dual exchange relations will be different than it was for the original relations (4.35-38). We shall not try to invert directly the dual symplectic form $\tilde{\Omega}_\infty$ because the fact that it is not defined everywhere on $G_L \times A_-$ makes the task more involved than the inversion of the form Ω_∞ in Sec 4.5. Instead, we shall proceed indirectly, by using the duality antisymplectomorphism $U : M_\infty \to \tilde{M}_\infty$ and the Poisson-Lie symmetry of $(M_\infty, \Omega_\infty)$. Indeed, we take into account the definition (4.41) of the diffeomorphism U to see that, for finding the dual brackets $\{\tilde{a}, \tilde{k}(\sigma)\}_\infty$, $\{\tilde{k}(\sigma) \overset{\otimes}{,} \tilde{k}(\sigma')\}_\infty$, it is sufficient to calculate the brackets $\{a^{-1}, \Xi_R^{-1}(ka)(\sigma)\}_\infty$, $\{\Xi_R^{-1}(ka)(\sigma) \overset{\otimes}{,} \Xi_R^{-1}(ka)(\sigma')\}_\infty$ with the help of the original Poisson bivector (4.34).

Consider an element $(k,a) \in G_R \times A_+ = M_\infty$ and write it as $ka = \Lambda_L(ka)\Xi_R^{-1}(ka)$. Thus we have

$$\{a^{-1}, \Xi_R^{-1}(ka)(\sigma)\}_\infty = \{a^{-1}, \Lambda_L^{-1}(ka)(\sigma)k(\sigma)a\}_\infty, \qquad (4.48)$$

$$\{\Xi_R^{-1}(ka)(\sigma) \overset{\otimes}{,} \Xi_R^{-1}(ka)(\sigma')\}_\infty = \{\Lambda_L^{-1}(ka)(\sigma)k(\sigma)a \overset{\otimes}{,} \Lambda_L^{-1}(ka)(\sigma')k(\sigma')a\}_\infty. \qquad (4.49)$$

If we take into account the following obvious Poisson matrix relations

$$\{AB \overset{\otimes}{,} CD\} = (A \otimes 1)\{B \overset{\otimes}{,} C\}(1 \otimes D) +$$

$$+ (A \otimes C)\{B \overset{\otimes}{,} D\} + \{A \overset{\otimes}{,} C\}(B \otimes D) + (1 \otimes C)\{1 \overset{\otimes}{,} D\}(B \otimes 1), \quad (4.50)$$

$$\{A \overset{\otimes}{,} B^{-1}\} = -(1 \otimes B^{-1})\{A \overset{\otimes}{,} B\}(1 \otimes B^{-1}), \qquad (4.51)$$

we realize that the dual exchange relations (4.48) and (4.49) can be worked out from the matrix Poisson brackets of the type $\{a \overset{\otimes}{,} ka\}_\infty$, $\{ka \overset{\otimes}{,} ka\}_\infty$, $\{\Lambda_L(ka) \overset{\otimes}{,} \Lambda_L(ka)\}_\infty$, $\{a \overset{\otimes}{,} \Lambda_L(ka)\}_\infty$ and $\{ka \overset{\otimes}{,} \Lambda_L(ka)\}_\infty$. Let us argue that we already know explicit formulae for all five brackets just listed. Indeed, the first two brackets in the list are given in Eqs. (4.35) and (4.36). Next, the fact that $\Lambda_L \circ \chi : (M_\infty, \Pi^\infty) \to (G^*, \Pi^*_{op})$ is the Poisson map implies that the third bracket is given by the ∞-current algebra relation (4.11) (cf. also (4.12) and (4.13)):

$$\{\Lambda_L(ka)(\sigma) \overset{\otimes}{,} \Lambda_L(ka)(\sigma')\}_\infty = \left(r + C\mathrm{cotg}\frac{\sigma - \sigma'}{2}\right) \times$$

$$\times \Big(\Lambda_L(ka)(\sigma) \otimes \Lambda_L(ka)(\sigma')\Big) - \Big(\Lambda_L(ka)(\sigma) \otimes \Lambda_L(ka)(\sigma')\Big)\left(r + C\mathrm{cotg}\frac{\sigma - \sigma'}{2}\right). \qquad (4.52)$$

Finally, we notice that the brackets $\{a \overset{\otimes}{,} \Lambda_L(ka)\}_\infty$, $\{ka \overset{\otimes}{,} \Lambda_L(ka)\}_\infty$ can be worked out as follows:

Consider the vector field v_T corresponding to the action of $T \in \mathcal{G}_R$ on M_∞. Corollary of Theorem 6 states that

$$< v_T, df >= \Pi^\infty(df, (\Lambda_L \circ \chi)^*(\rho, T)_\mathcal{D}),$$

for every function f defined on M_∞. Let Υ be the representation of the group K introduced in Sec 4.2 and Tr_Υ be the trace in this representation normalized in such a way that

$$Tr_\Upsilon(\Upsilon(A)\Upsilon(B)) = (A, B)_\mathcal{K}, \quad A, B \in \mathcal{K}^{\mathbf{C}}.$$

The right-invariant Maurer-Cartan form ρ on G^* can be written with the help of the matrix valued functions Υ defined in Sec 4.2:

$$\Upsilon(\rho) = d\Upsilon\Upsilon^{-1}, \tag{4.53}$$

Thus, using (4.3) and (4.4), we can write

$$< v_T, df >= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma Im Tr_\Upsilon(\Upsilon(T), \{f, (\Lambda_L \circ \chi)^* \Upsilon_\sigma\}_\infty(\Lambda_L \circ \chi)^* \Upsilon_\sigma^{-1}). \tag{4.54}$$

Let T_R^i a basis of $\mathcal{G}_R$, t_i the basis of $\mathcal{G}^*$ verifying the condition (3.7) and $k \in G_R$. We wish to express the quantity $\Upsilon_\sigma(T_R^i)\Upsilon_\sigma(k) \otimes \Upsilon_{\sigma'}(t_i)$, by using the Poisson brackets on M_∞. By invoking Eqs. (4.10), (4.54) and the notation of the proof of Corollary of Theorem 6, we have

$$\Upsilon_\sigma(T_R^i)\Upsilon_\sigma(k) \otimes \Upsilon_{\sigma'}(t_i) =$$

$$=< v_{T_R^i}, d\Upsilon_\sigma(k) > \otimes \Upsilon_{\sigma'}(t_i) = \{\Upsilon_\sigma(k) \overset{\otimes}{,} \Upsilon_{\sigma'}(\Lambda_L(ka))\}_\infty \Big(1 \otimes \Upsilon_{\sigma'}(\Lambda_L^{-1}(ka))\Big),$$

By suppressing the symbol Υ, as usual, we can thus write

$$\{k(\sigma) \overset{\otimes}{,} \Lambda_L(ka)(\sigma')\}_\infty =$$

$$= (T_R^i(\sigma) \otimes t_i(\sigma'))(k(\sigma) \otimes \Lambda_L(ka)(\sigma')) = (r + C \cot g \frac{\sigma - \sigma'}{2})(k(\sigma) \otimes \Lambda_L(ka)(\sigma')). \tag{4.55}$$

Note that we have evaluated the expression $T_R^i(\sigma) \otimes t_i(\sigma')$ in the basis $T_R^i = (T_R^\mu, B_R^{\hat\alpha}, C_R^{\hat\alpha})$, $\hat\alpha > 0$ and $t_i = (t_\mu, b_{\hat\alpha}, c_{\hat\alpha})$, $\hat\alpha > 0$, defined in Sec 4.1.

We know that the coordinates ϕ^μ are invariant with respect to the action of the $\mathcal{G}_R$ Poisson-Lie symmetry on M_∞, which, combined with Eqs. (4.29) and (4.53), gives

$$\{\phi^\mu, \Lambda_L(ka)\}_\infty \Lambda_L^{-1}(ka) = 0.$$

From this equality we finally infer

$$\{a \overset{\otimes}{,} \Lambda_L(ka)\}_\infty = 0. \tag{4.56}$$

We are now ready to write down the seeken dual exchange relations. We take into account Eqs. (4.35), (4.36), (4.52), (4.55), (4.56) as well as Eqs. (4.42), (4.50), (4.51) to find out

$$\{\tilde{a}, \tilde{k}(\sigma)\}_\infty = -i(\tilde{a} \otimes \tilde{k}(\sigma))(H^\mu \otimes H^\mu), \tag{4.57}$$

$$\{\tilde{k}(\sigma) \overset{\otimes}{,} \tilde{k}(\sigma')\}_\infty =$$

$$= (\tilde{k}(\sigma) \otimes \tilde{k}(\sigma'))(r(\tilde{a}^{-1}) + C\cot g\frac{\sigma - \sigma'}{2}) + (r + C\cot g\frac{\sigma - \sigma'}{2})(\tilde{k}(\sigma) \otimes \tilde{k}(\sigma')). \tag{4.58}$$

We remind that

$$\tilde{a} = a^{-1} = e^{-\phi^\mu H^\mu}, \quad \tilde{k} = \Lambda_L^{-1}(ka)ka$$

and the minus sign distinguishing the symplectic forms Ω_∞ and $U^*\tilde\Omega_\infty$ has been taken into account.

We stress that the dual $(\tilde{M}_\infty, \tilde\Omega_\infty)$-description of the $q \to \infty$ chiral phase space is U-equivalent to the original $(M_\infty, \Omega_\infty)$-description, in particular, the dual G_L-exchange relations (4.57), (4.58) are equivalent to the G_R-exchange relations (4.35-38). From the symmetry poin of view, the $(M_\infty, \Omega_\infty)$-formalism is better adapted to the explicit description of the $\mathcal{G}_R$-symmetry while the $(\tilde{M}_\infty, \tilde\Omega_\infty)$-formalism is better adapted to the explicit description of the $\mathcal{G}_L$-symmetry. Note, however, one important point: the submanifold $\tilde{M}_\infty \subset G_L \times A_-$ is not invariant under the global G_L-action while the manifold $M_\infty = G_R \times A_+$ is invariant under the global G_R-action. Said in other words, the $\mathcal{G}_R$-symmetry is global while the $\mathcal{G}_L$-symmetry is only local. This is the

reason why we have constructed our exposition starting from the $(\mathcal{M}_\infty, \Omega_\infty)$-description where the global G_R-symmetry looks naturally. However, the $(\tilde{M}_\infty, \tilde{\Omega}_\infty)$-description has also its assets because, as we are going to see, it is better adapted for the comparison of the finite q chiral WZW model with the $q \to \infty$ chiral WZW model.

Now we wish to compare the dual exchange relations (4.57) and (4.58) with the exchange relations of the finite q chiral WZW model of Ref. [9]. To do that we should take into account that our parameter q is the parameter $1/q$ of Ref.[9], as it follows from the comparison of Eq. (4.5) of the present paper and of Eq. (4.113) of [9]. In order not to cause confusion, we shall denote by q' the parameter q of [9] and we set $\varepsilon' = \ln q'$. The exchange relations for finite q' were explicited in Eqs. (5.159) and (5.161) of Ref. [9] only for $q' > 1$ (or $\varepsilon' > 0$) and we have to make them explicit also for $0 < q' < 1$ (since when our q goes to ∞ the "old" q' of [9] goes to 0^+). By reusing the method of Sec 5.2.4 of [9], we obtain for $\varepsilon' < 0$:

$$\{\tilde{a}, \tilde{k}(\sigma)\}_{\varepsilon'} = -i\varepsilon'(\tilde{a} \otimes k(\sigma))(H^\mu \otimes H^\mu), \tag{4.59}$$

$$\{\tilde{k}(\sigma) \overset{\otimes}{,} \tilde{k}(\sigma')\}_{\varepsilon'} =$$

$$= (\tilde{k}(\sigma) \otimes \tilde{k}(\sigma'))\hat{r}_{\varepsilon'}(\tilde{a}, \sigma - \sigma') + \varepsilon'(r + C \mathrm{cotg}\frac{\sigma - \sigma'}{2})(\tilde{k}(\sigma) \otimes \tilde{k}(\sigma')), \tag{4.60}$$

where $\tilde{k}(\sigma) \in G_L$ and $\hat{r}_{\varepsilon'}(\tilde{a}, \sigma)$ is the Felder elliptic dynamical r-matrix [7] given by

$$\hat{r}_{\varepsilon'}(\tilde{a}, \sigma) =$$

$$= \frac{\varepsilon'}{\pi}\rho(\frac{\sigma}{2\pi}, -\frac{ik\varepsilon'}{\pi})H^\mu \otimes H^\mu + \frac{\varepsilon'}{\pi}\sum_{\alpha \in \Phi}\frac{|\alpha|^2}{2}\sigma_{-\frac{\varepsilon' k a^\mu \langle \alpha, H^\mu \rangle}{\pi i}}(\frac{\sigma}{2\pi}, -\frac{ik\varepsilon'}{\pi})E^\alpha \otimes E^{-\alpha}.$$

We remind that here $\varepsilon' < 0$, $\tilde{a} = e^{k\varepsilon' a^\mu H^\mu}$, k is the level of the q-WZW model and the coordinates a^μ parametrize the so called positive Weyl alcove $\mathcal{A}_+^1$ (cf. Sec 5.2 of [9]). Using the classical formulae from Ref. [20] which define the elliptic functions $\sigma_{-y}(z, \tau)$, $\rho(z, \tau)$

$$\sigma_{-y}(z, \tau) = \pi(\mathrm{cotg}\pi z + \mathrm{cotg}\pi y) + 4\pi\Sigma_{m,n>0}e^{2\pi i \tau mn}\sin 2\pi(mz + ny);$$

$$\rho(z, \tau) = \pi\mathrm{cotg}\pi z + 4\pi\Sigma_{n>0}\frac{e^{2\pi in\tau}\sin 2\pi nz}{1 - e^{2\pi in\tau}},$$

36

and keeping fixed the expression $\phi^\mu = -k\varepsilon' a^\mu$, we find that

$$\lim_{\varepsilon' \to -\infty} \frac{1}{\varepsilon'} \hat{r}_{\varepsilon'}(\tilde{a}, \sigma) = \left(r(e^{\phi^\mu H^\mu}) + C\mathrm{cotg} \frac{\sigma - \sigma'}{2} \right),$$

This gives, in turn, for $\tilde{a} = e^{-\phi^\mu H^\mu}$

$$\lim_{\varepsilon' \to -\infty} \frac{1}{\varepsilon'} \{\tilde{a}, \tilde{k}(\sigma)\}_{\varepsilon'} = -i(\tilde{a} \otimes k(\sigma))(H^\mu \otimes H^\mu),$$

$$\lim_{\varepsilon' \to -\infty} \frac{1}{\varepsilon'} \{\tilde{k}(\sigma), \tilde{k}(\sigma')\}_{\varepsilon'} =$$

$$= (\tilde{k}(\sigma) \otimes \tilde{k}(\sigma'))(r(\tilde{a}^{-1}) + C\mathrm{cotg} \frac{\sigma - \sigma'}{2}) + (r + C\mathrm{cotg} \frac{\sigma - \sigma'}{2})(\tilde{k}(\sigma) \otimes \tilde{k}(\sigma')).$$

We have just established, as expected, that $\lim_{\varepsilon' \to -\infty} \frac{1}{\varepsilon'} \{.,.\}_{\varepsilon'} = \{. \overset{\otimes}{,} .\}_\infty$, or, in other words, the finite q chiral WZW exchange relations (4.59), (4.60) give in the $q \to \infty$ limit the (dual) G_L-exchange relations (4.57),(4.58) of the chiral ∞-WZW model.

The reader may feel intrigued why we did not start our exposition of the ∞-WZW model by first establishing the $q \to \infty$ limit of the chiral exchange relations, but, instead, we have first exposed the theory of the affine Poisson groups etc. Actually, the more "theoretical" approach, that we have chosen, have evident benefits. The most important is that it allowed us to identify the singularities of the dual exchange relations (4.57) and (4.58). Indeed, the following phenomenon takes place: the chiral WZW phase space for finite q is just the manifold $G_L \times A^1$, where A^1 is the (compact) subset of A obtained by exponentiating the positive Weyl alcove as follows: $e^{k\varepsilon' a^\mu H^\mu} \in A^1$. In the $q \to \infty$ limit (or the $\varepsilon' \to -\infty$ limit), the set A^1 gets expanded to the exponentiated negative Weyl chamber $A_- = \exp \mathcal{A}_-$ and the chiral Poisson structure, defined on the *whole* manifold $G_L \times A_-$ by the brackets (4.57) and (4.58), becomes invertible only on the *submanifold* $\tilde{M}_\infty \subset G_L \times A_-$. Our approach, using the theory of affine Poisson groups, gave as the natural characterization of the regular submanifold $\tilde{M}_\infty$ in terms of the domain of definition of the maps Ξ_L and Λ_R.

The general theory of the affine Poisson groups has helped us to clarify one more subtle point. Indeed, as we have already mentioned, there is a price to pay for the restriction to the regular submanifold $\tilde{M}_\infty \subset G_L \times A_-$, namely,

the loop group $G_L = L_{pol}K$ no longer acts on $\tilde{M}_\infty$. At the first sight, this may look bad, because the $q \to \infty$ limit seems to deprive the WZW model from its interesting symmetry structure, however, as we have again learned from Theorems 2 and 7, the G_L-symmetry does survive the limit in its local $\mathcal{G}_L$-form. On the top of that, there are further added benifits of our approach: we have discovered the new G_R symmetry, which emerged in the $q \to \infty$ limit and also the remarkable $\mathcal{G}_L \leftrightarrow \mathcal{G}_R$ duality of the chiral ∞-WZW model.

4.8 $q \to \infty$ Hamiltonian

A dynamical system is a triple (M, Ω, E) where M is a phase space, Ω a symplectic form on it and E is a one-parameter group of symplectomorphisms of M defining the time evolution. We have often spoken about a particular dynamical system called the chiral ∞-WZW model but, so far, we have specified only its phase space $M_\infty = G_R \times A_+$ and its symplectic form Ω_∞ given by Eq. (4.17). In this section, we shall fill the gap and define also the one-parameter group E_∞ describing the time evolution in the chiral ∞-WZW model. In order to do that we first define a suitable parametrization of the phase space M_q, for a finite q.

For finite q (including the standard non-deformed $q = 1$ case), the time evolution is most easily described in the so called "monodromic" variables. Let us define them. The phase space M_q of the chiral q-WZW model is the direct product $L_{pol}K \times \mathcal{A}^1$, where $\mathcal{A}^1$ is the (compact) Weyl alcove. The usual periodic parametrization of $L_{pol}K \times \mathcal{A}^1$ is $(\tilde{k}(\sigma), a^\mu H^\mu)$. In the monodromic parametrization, the points in the phase space M_q are quasi-periodic maps $m : \mathbf{R} \to K$, fulfilling the monodromy condition

$$m(\sigma + 2\pi) = m(\sigma)M, \quad M = \exp\left(-2\pi i a^\mu H^\mu\right).$$

The transformation from the monodromic parametrization into the periodic one is given by the relations

$$\exp\left(-2\pi i a^\mu H^\mu\right) = m^{-1}(\sigma)m(\sigma + 2\pi), \quad \tilde{k}(\sigma) = m(\sigma)\exp\left(i a^\mu H^\mu \sigma\right).$$

The time evolution E_q of the finite q chiral WZW model is defined very simply: a point $m(\sigma) \in M_q$ at the time $\tau = 0$ gets evolved to the point $m(\sigma - \tau)$ at the time τ. In the periodic parametrization, this time evolution

gets translated into

$$\tilde{k}(\sigma) \to \tilde{k}(\sigma - \tau) \exp\left(ia^\mu H^\mu \tau\right), \quad a^\mu \to a^\mu. \tag{4.61}$$

It is not difficult to see that the transformation (4.61) defines a symplecto-morphism. This follows from the fact that the finite q exchange relations (4.59) and (4.60) (which completely characterize the symplectic structure on M_q) are invariant with respect to (4.61).

Coming back to the case $q \to \infty$, we define the time evolution in terms of the dual variables $(\tilde{k}, \tilde{a}) \in \tilde{M}_\infty$ (cf. Eq. (4.41)). The advantage of working with the dual variables in this context is clear: they are the best adapted for the description of the $q \to \infty$ limit. We remind that $\tilde{a} = e^{k\varepsilon' a^\mu H^\mu} = e^{-\phi^\mu H^\mu}$, which gives $a^\mu = -\frac{\phi^\mu}{\varepsilon' k}$. Thus the E_q evolution transformation (4.61) can be rewritten as

$$\tilde{k}(\sigma) \to \tilde{k}(\sigma - \tau) \exp\left(-\frac{i\phi^\mu}{\varepsilon' k} H^\mu \tau\right), \quad \phi^\mu \to \phi^\mu.$$

The $\varepsilon' \to -\infty$ limit is achieved by keeping ϕ^μ fixed which suggests the following time evolution E_∞ of the chiral ∞-WZW model:

$$\tilde{k}(\sigma) \to \tilde{k}(\sigma - \tau), \quad \phi^\mu \to \phi^\mu. \tag{4.62}$$

Let us verify, that the suggested time evolution (4.62) is correctly defined, which means that it leaves invariant the subspace $\tilde{M}_\infty \subset L_{pol}K \times A_-$ and it defines the one-parameter group of symplectomorphisms of $\tilde{M}_\infty$. The latter statement follows from the obvious invariance of the dual exchange relations (4.57), (4.58) with respect to the transformation (4.62). To prove the former statement is also easy. Indeed, let $l(\sigma)$ be an element of $L_{pol}K^{\mathbf{C}}$ which is also in the domain of definition of the maps Λ_R and Ξ_L, i.e. $l(\sigma) \in S_\infty$. Then it follows that also $l(\sigma - \tau)$ is in $S_\infty = G_R G^*$, since the transformation of $D = L_{pol}K^{\mathbf{C}}$ defined by $l(\sigma) \to l(\sigma - \tau)$ leaves the subgroups G_R and G^* invariant.

In this section, we have completed the definition of the chiral ∞-WZW model by defining the consistent time evolution E_∞ on the phase space $(\tilde{M}, \tilde{\Omega}_\infty)$. However, we may ask another question: is this evolution generated by a Hamiltonian function? The answer to this question is affirmative provided that we enlarge slightly the phase space M_∞ by working with the smooth loop

39

groups $LK^{\mathbf{C}}$, LK etc. rather than with the polynomial ones $L_{pol}K^{\mathbf{C}}$, $L_{pol}K$ etc. Now recall the result described in Sec 8.9 of [18] that every infinitesimal symplectomorphism of an infinite-dimensional simple connected symplectic manifold has a Hamiltonian function. Let us therefore argue that the chiral ∞-WZW phase space $\tilde{M}_\infty$ is simply connected. First of all, we know that $\tilde{M}_\infty$ is diffeomorphic to $M_\infty = G_R \times A_-$ which gives the following relation between the fundamental groups:

$$\pi_1(\tilde{M}_\infty) = \pi_1(G_R) \times \pi_1(A_-).$$

Obviously, A_- is simply connected and it remains to show that G_R is simply connected.

Denote respectively by $L^0K^{\mathbf{C}}$ and L^0K the subgroups of $LK^{\mathbf{C}}$ and LK formed of the loops verifying $l(0) = e_{K\mathbf{C}}$ and $k(0) = e_K$. Similarly, denote by G_R^0 the subgroup of G_R for which $\gamma_0 = e_{K\mathbf{C}}$ (cf. the expansion (4.2)). Clearly, $LK^{\mathbf{C}}$ is diffeomorphic to $K^{\mathbf{C}} \times L^0K^{\mathbf{C}}$, LK is diffeomorphic to $K \times L^0K$ and G_R is diffeomorphic to $K \times G_R^0$. Thus we obtain

$$\pi_1(LK^{\mathbf{C}}) = \pi_1(K^{\mathbf{C}}) \times \pi_1(L^0K^{\mathbf{C}}) = \pi_1(K^{\mathbf{C}}) \times \pi_2(K^{\mathbf{C}}), \qquad (4.63)$$

$$\pi_1(LK) = \pi_1(K) \times \pi_1(L^0K) = \pi_1(K) \times \pi_2(K), \qquad (4.64)$$

$$\pi_1(G_R) = \pi_1(K) \times \pi_1(G_R^0). \qquad (4.65)$$

Note also that the global decomposition $LK^{\mathbf{C}} = G^*LK$ (cf. Lemma 6) implies also a global decomposition $LK^{\mathbf{C}} = G_0^*(AN)LK$, where G_0^* is the subgroup of G^* for which $\gamma_0 = e_{K\mathbf{C}}$. Taking the hermitian conjugation of the global decomposition $LK^{\mathbf{C}} = G_0^*(AN)LK$, we establish that $LK^{\mathbf{C}}$ is diffeomorphic to $LK \times (AN)^\dagger \times G_R^0$. Hence we have

$$\pi_1(LK^{\mathbf{C}}) = \pi_1(G_R^0) \times \pi_1((AN)^\dagger)) \times \pi_1(LK). \qquad (4.66)$$

We have supposed that K is simple connected and simply connected, which means that the fundamental groups $\pi_1(K)$, $\pi_1(K^{\mathbf{C}})$ and $\pi_1((AN)^\dagger)$ are trivial. Moreover, the classical theorem says (cf. Sec 8.6 of [18]) that the second homotopy groups $\pi_2(K)$ and $\pi_2(K^{\mathbf{C}})$ are also trivial. Thus we deduce from the relations (4.63-66) the desired result that the fundamental group of G_R is also trivial.

We have established the existence of the hamiltonian generating the simple time evolution (4.62), however, we do not have any explicit formula expressing it as the function of $\tilde{k}$ and $\tilde{a}$. This type of situation took place also for the finite q case; it was only for the standard $q = 1$ WZW model that such an explicite (i.e. Sugawara) formula could have been written.

5 Conclusions and outlook

The paper is devoted to the explicit construction of the dynamical system, which can be interpreted as the $q \to \infty$ limit of the q-WZW model. The most important feature of this system is its symmetry structure; in fact, the limiting model has two *non-isomorphic* symmetry groups G_L and G_R, which is not true for the finite q WZW model (including the standard case $q = 1$), thus we may even say that the symmetry structure of the $q \to \infty$ WZW model is richer and more intriguing than that of its finite q counterpart. In order to identify the symmetries of the model, we have used in its full richness the concept of the affine Poisson group and of the anomalous Poisson-Lie symmetry. In fact, the formulation of the $q \to \infty$ WZW model based on the notion of the symplectic grupoid of the affine Poisson group was very insightful; in particular, we would not have suspected the existence of the remarkable duality transformation (4.41) if we did not take the inspiration from Lemma 5.

There are also aspects of the structure of the $q \to \infty$ WZW model which did not follow from the general theory of the affine Poisson groups, like e.g. the chiral decomposability. The possibility to study the simpler chiral $q \to \infty$ WZW model, which also enjoys the anomalous Poisson-Lie symmetry, has lead to very explicit formulae like the exchange relations (4.57) and (4.58) and was also very suggestive on general grounds. In particular, we found remarkable that the $q \to \infty$ trigonometric exchange relations (4.57) and (4.58) are so much simpler than the finite q elliptic exchange relations (4.59) and (4.60).

Finally, we believe that the quantization of the results of the present paper will be doable due to the simplicity of the trigonometric exchange relations (4.57) and (4.58). The eventual quantization will probably make a fruitful use of the theory of twisting of Hopf algebras developed by Majid [17].

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